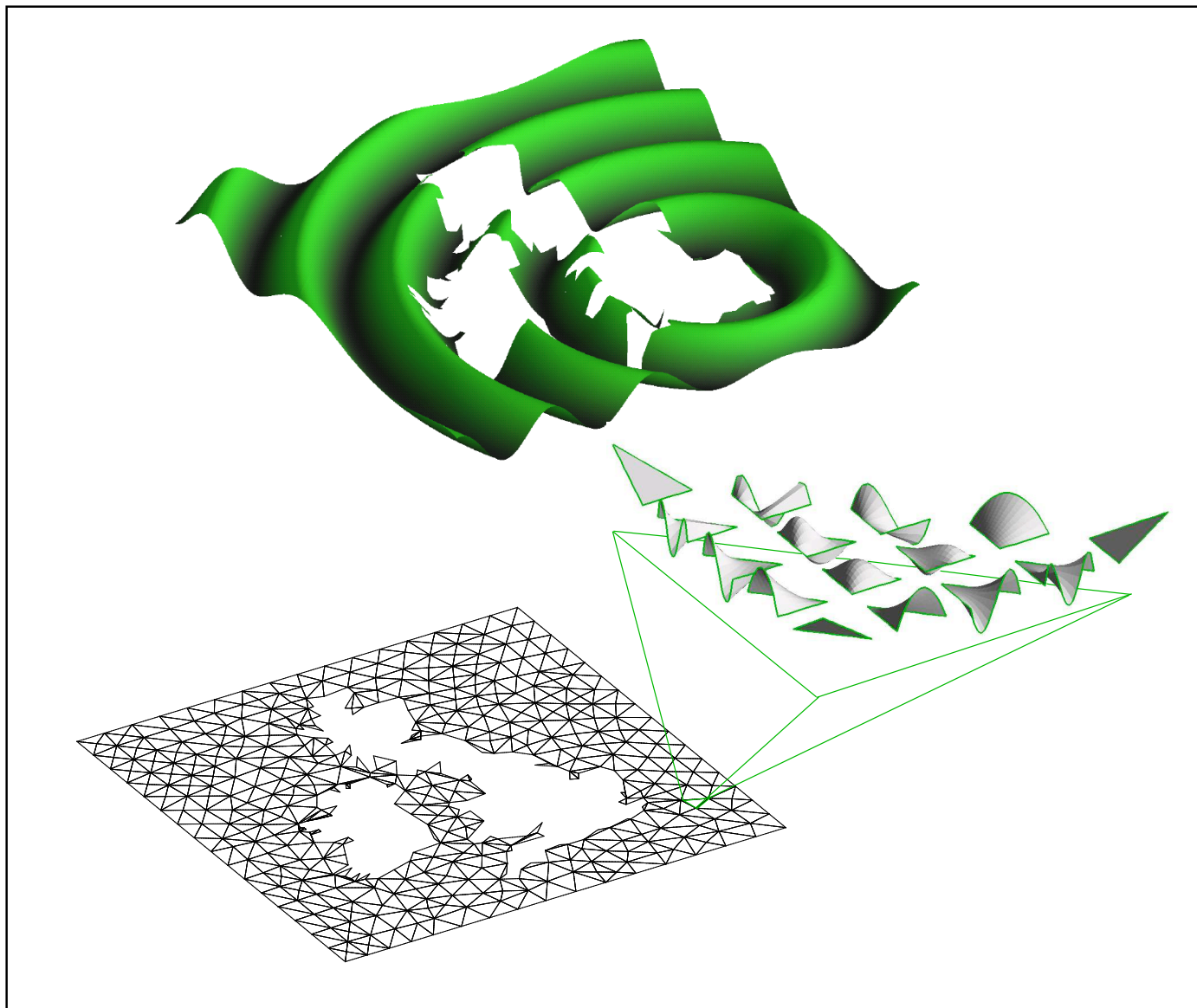
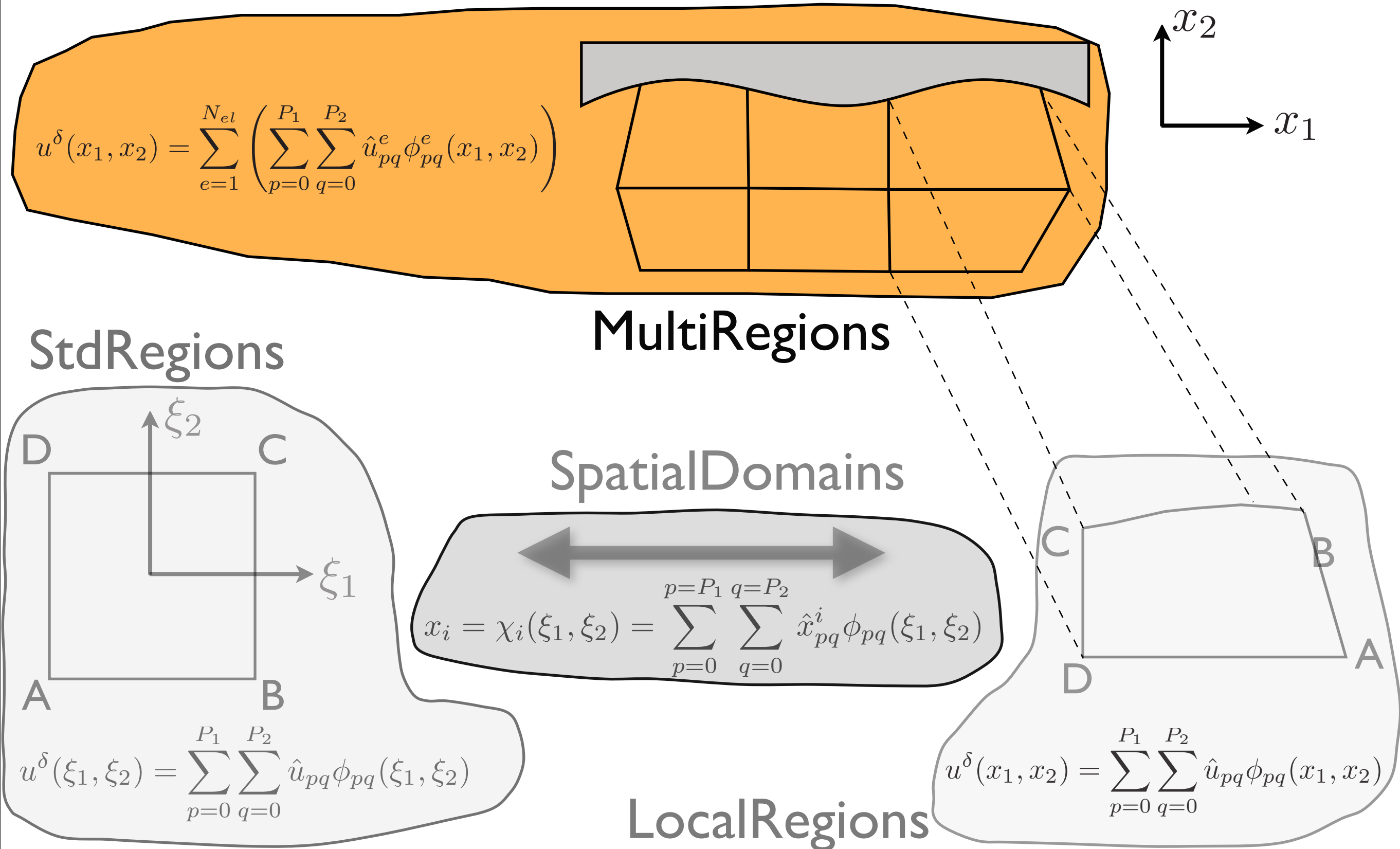


Expansions in Multiple Regions



The big picture



Outline

- 1D formulation of Helmholtz Problem
 - Elemental formulation
 - Global assembly
- 2D Formulation of Helmholtz problem
 - Global assembly

1	Fundamental Concepts in One Dimension	1
1.1	Method of Weighted Residuals	2
1.2	Galerkin Formulation	5
1.2.1	Descriptive Formulation	5
1.2.2	Two-Domain Linear Finite Element Example	8
1.2.3	Mathematical Formulation	11
1.2.4	Mathematical Properties of the Galerkin Approximation	13
1.2.5	Residual Equation for C^0 Test and Trial Functions	15
1.3	One-Dimensional Expansion Bases	16
1.3.1	Elemental Decomposition: The h -Type Extension	16
1.3.2	Polynomial Expansions: The p -Type Extension	22
1.3.3	Modal Polynomial Expansions	28
1.3.4	Nodal Polynomial Expansions	31
1.4	Elemental Operations	34
1.4.1	Numerical Integration	34
1.4.2	Differentiation	39
1.5	Error Estimates	42
1.5.1	h -Convergence of Linear Finite Elements	42
1.5.2	L^2 Error of the p -Type Interpolation in a Single Element	44
1.5.3	General Error Estimates for hp Elements	45
1.6	Implementation of a 1D Spectral/ hp Element Solver:	46
1.6.1	Exercises	46
1.6.2	Convergence Examples	50
3	Multi-dimensional Formulation	98
3.1	Local Elemental Operations	99
3.1.1	Integration within the Standard Region Ω_{st}	99
3.1.2	Differentiation in the Standard Region Ω_{st}	104
3.1.3	Operations within General Shaped Elements	109
3.1.4	Discrete Evaluation of the Surface Jacobian	115
3.1.5	Elemental Projections and Transformations	118
3.1.6	Sum Factorisation/Tensor Product Operations	129
3.2	Global Operations	134
3.2.1	Global Assembly and Connectivity	135
3.2.2	Global Matrix System	146
3.2.3	Static Condensation/Substructuring	149
3.2.4	Global Boundary System Numbering and Ordering to Enforce Dirichlet Boundary Conditions	153
3.3	Pre- and Post-Processing Issues	157
3.3.1	Boundary Condition Discretisation	158
3.3.2	Elemental Boundary Transformation	158
3.3.3	Mesh Generation for Spectral/ hp Element Discretisation	161
3.3.4	Global Coarse Meshing	162
3.3.5	High-Order Mesh Generation	163
3.3.6	Particle Tracking in Spectral/ hp Element Discretisations	172
3.4	Exercises: Implementation of a 2D Spectral/ hp Element solver for a Global Projection Problem Using a C^0 Galerkin Formulation	181

Helmholtz problem

Poisson Equation: $\nabla^2 u - \lambda u = f$

ID: $\mathbb{L}(u) = \frac{\partial^2 u}{\partial x^2} - \lambda u + f = 0, \quad u(0) = g_{\mathcal{D}}, \quad \frac{\partial u}{\partial x}(l) = g_{\mathcal{N}}.$

Integral formulation (MWR): $\int_0^l v \frac{\partial^2 u}{\partial x^2} - \int_0^l \lambda v u \, dx + \int_0^l v f \, dx = 0.$

Integrate by parts: $\int_0^l v \frac{\partial^2 u}{\partial x^2} dx = \left[v \frac{\partial u}{\partial x} \right]_0^l - \int_0^l \frac{\partial v}{\partial x} \frac{\partial u}{\partial x} dx$

Weak formulation: $\int_0^l \frac{\partial v}{\partial x} \frac{\partial u}{\partial x} + \int_0^l \lambda v u \, dx = \int_0^l v f \, dx + \left[v \frac{\partial u}{\partial x} \right]_0^l$

Enforcing Neumann BC's:

$$\left[v(l) \frac{\partial u}{\partial x} \Big|_l - v(0) \frac{\partial u}{\partial x} \Big|_0 \right]$$

\downarrow $g_{\mathcal{N}}$ \downarrow 0

Helmholtz problem:

Imposing Dirichlet boundary conditions

ID: $\mathbb{L}(u) = \frac{\partial^2 u}{\partial x^2} - \lambda u + f = 0,$ $u(0) = g_{\mathcal{D}},$ $\frac{\partial u}{\partial x}(l) = g_{\mathcal{N}}.$

Weak formulation: $\int_0^l \frac{\partial v}{\partial x} \frac{\partial u}{\partial x} + \int_0^l \lambda v u \, dx = \int_0^l v f \, dx + \left[v \frac{\partial u}{\partial x} \right]_0^l$

Lift BC: $u^{\delta} = u^{\mathcal{D}} + u^{\mathcal{H}}$ $u^{\mathcal{D}}(0) = g_{\mathcal{D}}$ $u^{\mathcal{H}}(0) = 0$
(Homogenize problem)

$$\int_0^l \frac{\partial v^{\mathcal{H}}}{\partial x} \frac{\partial u^{\mathcal{H}}}{\partial x} dx + \lambda \int_0^l v^{\mathcal{H}} u^{\mathcal{H}} dx = \int v f^* dx$$
$$f^* = f - v(l)g_{\mathcal{N}} - \frac{\partial v^{\mathcal{D}}}{\partial x} \frac{\partial u^{\mathcal{D}}}{\partial x} + \lambda v^{\mathcal{H}} u^{\mathcal{H}}$$

Discrete Approximation

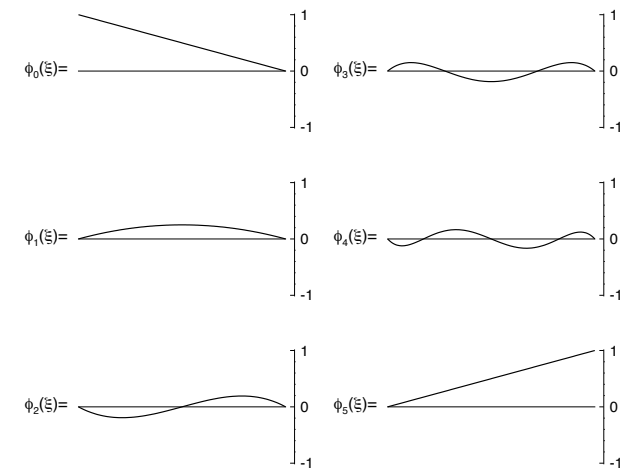
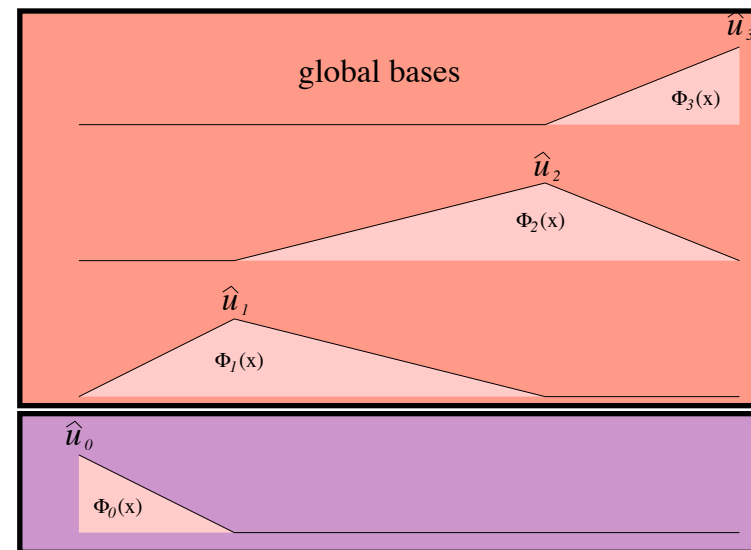
Global approximation - C^0

$$u \Rightarrow u^\delta = \sum \hat{u}_i \Phi_i(x)$$

$$v \Rightarrow v^\delta = \sum_i \hat{v}_i \Phi_i(x)$$

$$u^{\mathcal{D}}(0) = g_{\mathcal{D}}$$

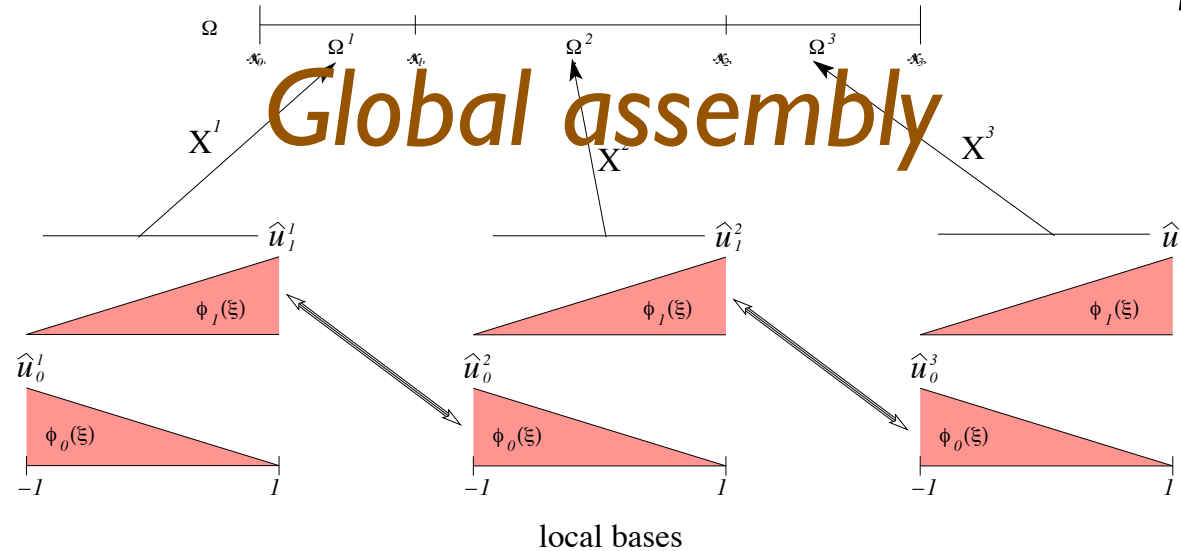
$$u^{\mathcal{H}}(0) = 0$$



Local P-modes

Local approximation

$$u^\delta = \sum_i \hat{u}_i \Phi_i(x) = \sum_e \sum_p^{nel} \hat{u}_p^e \phi_p(x)$$



$$\int_0^l \frac{\partial v^{\mathcal{H}}}{\partial x} \frac{\partial u^{\mathcal{H}}}{\partial x} dx + \lambda \int_0^l v^{\mathcal{H}} u^{\mathcal{H}} dx = \int v f^* dx$$

$$f^* = f - v(l)g_{\mathcal{N}} - \frac{\partial v^{\mathcal{D}}}{\partial x} \frac{\partial u^{\mathcal{D}}}{\partial x} + \lambda v^{\mathcal{H}} u^{\mathcal{H}}$$

Discrete spaces

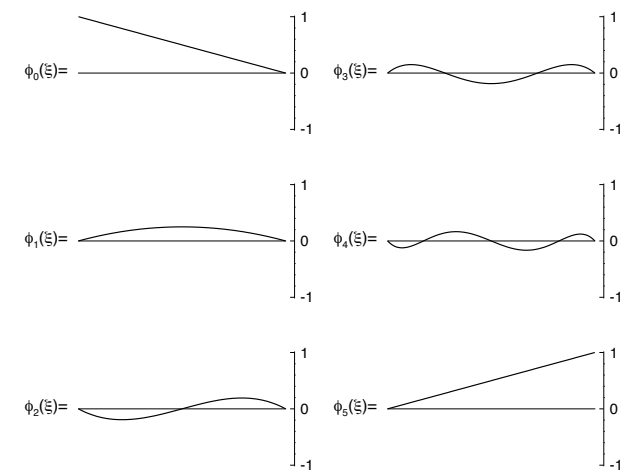
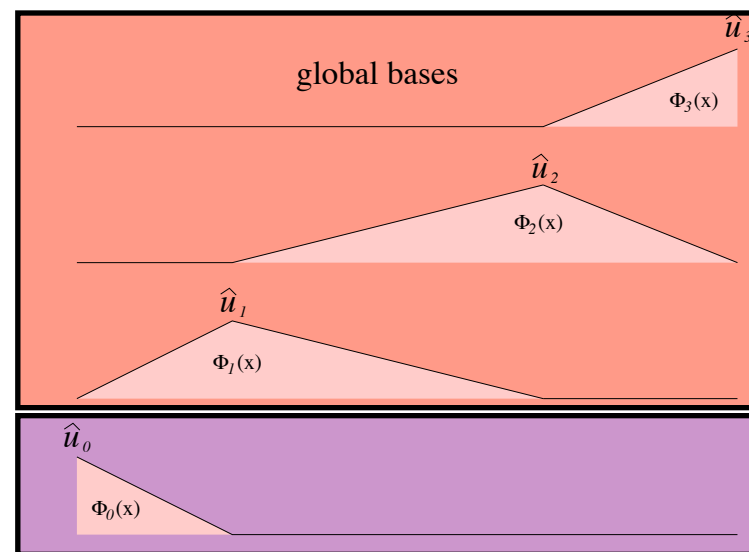
Global approximation - C^0

$$u \Rightarrow u^\delta = \sum \hat{u}_i \Phi_i(x)$$

$$v \Rightarrow v^\delta = \sum_i \hat{v}_i \Phi_i(x)$$

$$u^{\mathcal{D}}(0) = g_{\mathcal{D}}$$

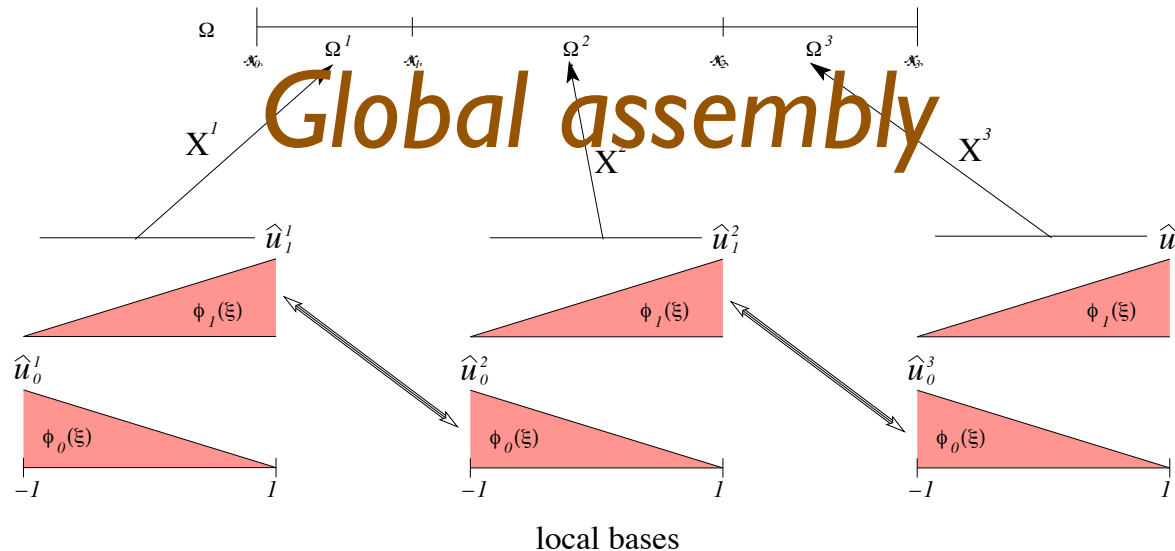
$$u^{\mathcal{H}}(0) = 0$$



Local P-modes

Local approximation

$$u^\delta = \sum_i \hat{u}_i \Phi_i(x) = \sum_e^{nel} \sum_p \hat{u}_p^e \phi_p(x)$$



$$\sum_i \hat{v}_j \left\{ \sum_j \int_0^l \left[\frac{\partial \Phi_i^{\mathcal{H}}}{\partial x} \frac{\partial \Phi_j^{\mathcal{H}}}{\partial x} + \lambda \Phi_i^{\mathcal{H}} \Phi_j^{\mathcal{H}} \right] \hat{u}_j dx = \int \Phi_i^{\mathcal{H}} f^* dx \right\}$$

$\mathbf{L}[i][j]$
 $\mathbf{M}[i][j]$
 $\mathbf{f}[i]$

Outline

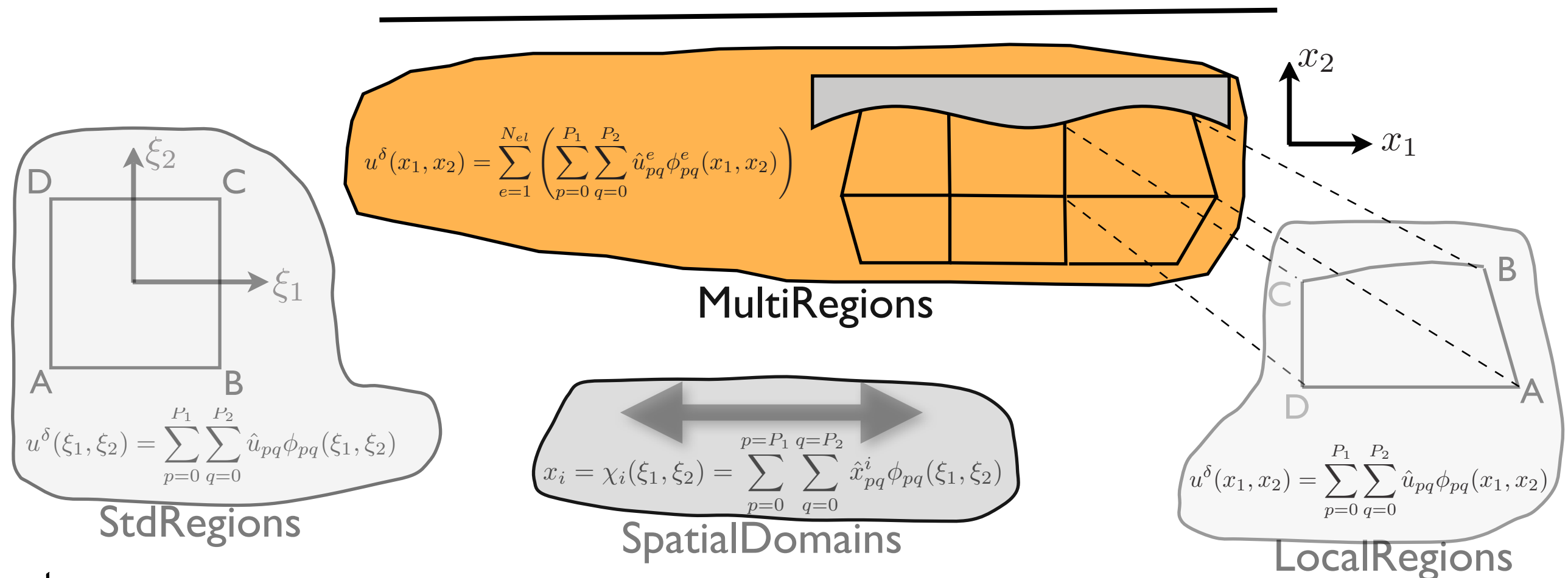
- 1D formulation of Helmholtz Problem
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1	Fundamental Concepts in One Dimension	1
1.1	Method of Weighted Residuals	2
1.2	Galerkin Formulation	5
1.2.1	Descriptive Formulation	5
1.2.2	Two-Domain Linear Finite Element Example	8
1.2.3	Mathematical Formulation	11
1.2.4	Mathematical Properties of the Galerkin Approximation	13
1.2.5	Residual Equation for C^0 Test and Trial Functions	15
1.3	One-Dimensional Expansion Bases	16
1.3.1	Elemental Decomposition: The h -Type Extension	16
1.3.2	Polynomial Expansions: The p -Type Extension	22
1.3.3	Modal Polynomial Expansions	28
1.3.4	Nodal Polynomial Expansions	31
1.4	Elemental Operations	34
1.4.1	Numerical Integration	34
1.4.2	Differentiation	39
1.5	Error Estimates	42
1.5.1	h -Convergence of Linear Finite Elements	42
1.5.2	L^2 Error of the p -Type Interpolation in a Single Element	44
1.5.3	General Error Estimates for hp Elements	45
1.6	Implementation of a 1D Spectral/ hp Element Solver:	46
1.6.1	Exercises	46
1.6.2	Convergence Examples	50
3	Multi-dimensional Formulation	98
3.1	Local Elemental Operations	99
3.1.1	Integration within the Standard Region Ω_{st}	99
3.1.2	Differentiation in the Standard Region Ω_{st}	104
3.1.3	Operations within General Shaped Elements	109
3.1.4	Discrete Evaluation of the Surface Jacobian	115
3.1.5	Elemental Projections and Transformations	118
3.1.6	Sum Factorisation/Tensor Product Operations	129
3.2	Global Operations	134
3.2.1	Global Assembly and Connectivity	135
3.2.2	Global Matrix System	146
3.2.3	Static Condensation/Substructuring	149
3.2.4	Global Boundary System Numbering and Ordering to Enforce Dirichlet Boundary Conditions	153
3.3	Pre- and Post-Processing Issues	157
3.3.1	Boundary Condition Discretisation	158
3.3.2	Elemental Boundary Transformation	158
3.3.3	Mesh Generation for Spectral/ hp Element Discretisation	161
3.3.4	Global Coarse Meshing	162
3.3.5	High-Order Mesh Generation	163
3.3.6	Particle Tracking in Spectral/ hp Element Discretisations	172
3.4	Exercises: Implementation of a 2D Spectral/ hp Element solver for a Global Projection Problem Using a C^0 Galerkin Formulation	181

Putting it all together

$$\sum_i \hat{v}_j \left\{ \sum_j \int_0^l \left[\frac{\partial \Phi_i^{\mathcal{H}}}{\partial x} \frac{\partial \Phi_j^{\mathcal{H}}}{\partial x} + \lambda \Phi_i^{\mathcal{H}} \Phi_j^{\mathcal{H}} \right] \hat{u}_j dx = \int \Phi_i^{\mathcal{H}} f^* dx \right\}$$

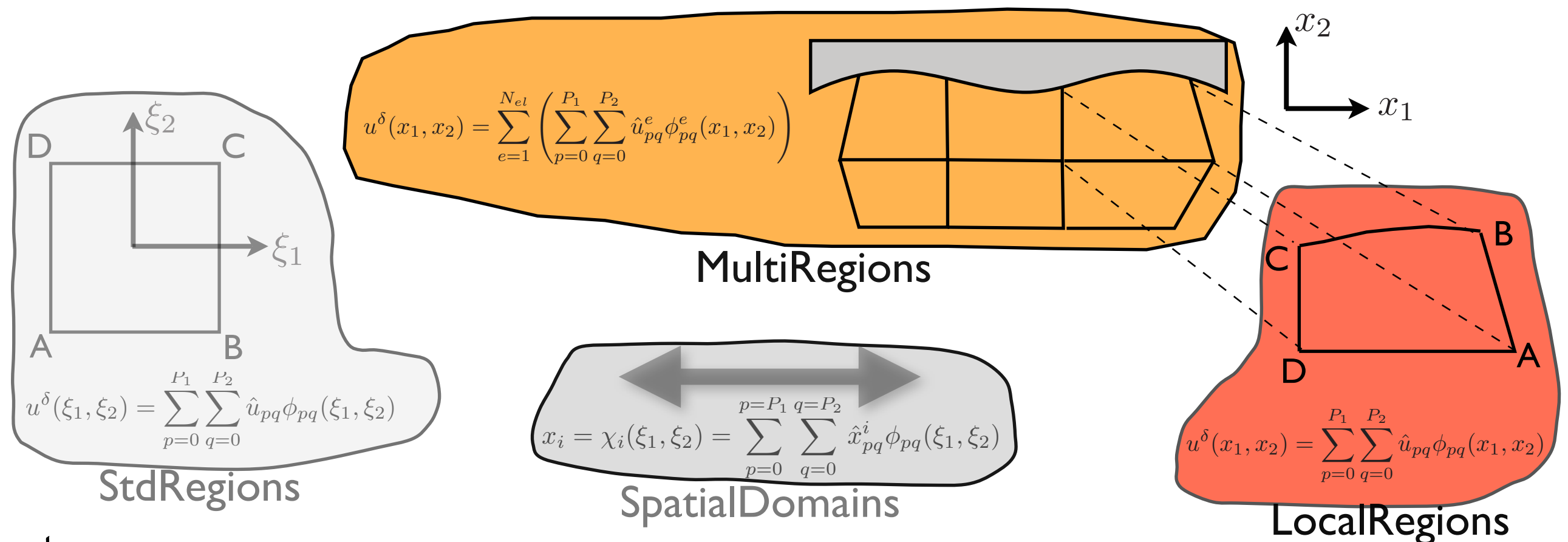
$$\mathbf{f}[i] = \int \Phi_i f^* dx$$



Putting it all together

$$\sum_i \hat{v}_j \left\{ \sum_j \int_0^l \left[\frac{\partial \Phi_i^{\mathcal{H}}}{\partial x} \frac{\partial \Phi_j^{\mathcal{H}}}{\partial x} + \lambda \Phi_i^{\mathcal{H}} \Phi_j^{\mathcal{H}} \right] \hat{u}_j dx = \int \Phi_i^{\mathcal{H}} f^* dx \right\}$$

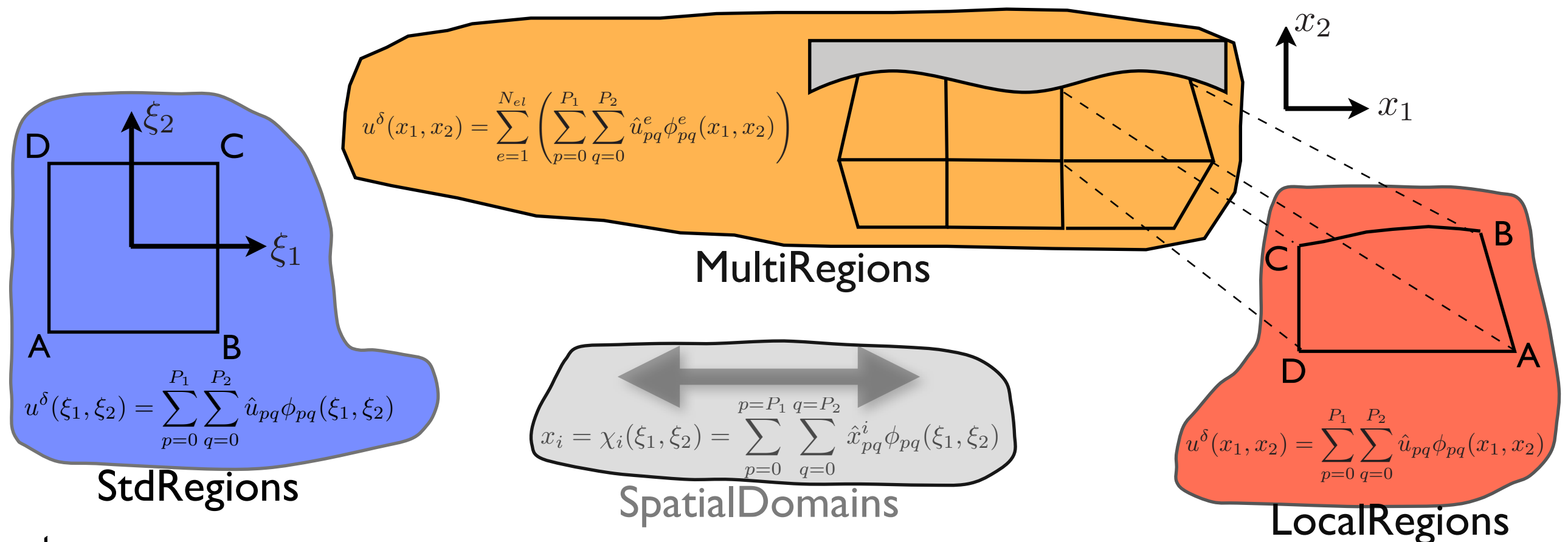
$$\mathbf{f}[i] = \int \Phi_i f^* dx = \sum_e^{nel} \sum_p \int_{\Omega^e} \phi_p(x) f^* dx = \sum_e^{nel} \sum_p \int_{\Omega^e} \phi_p(\chi^e(\xi)) f^* J^e d\xi$$



Putting it all together

$$\sum_i \hat{v}_j \left\{ \sum_j \int_0^l \left[\frac{\partial \Phi_i^{\mathcal{H}}}{\partial x} \frac{\partial \Phi_j^{\mathcal{H}}}{\partial x} + \lambda \Phi_i^{\mathcal{H}} \Phi_j^{\mathcal{H}} \right] \hat{u}_j dx = \int \Phi_i^{\mathcal{H}} f^* dx \right\}$$

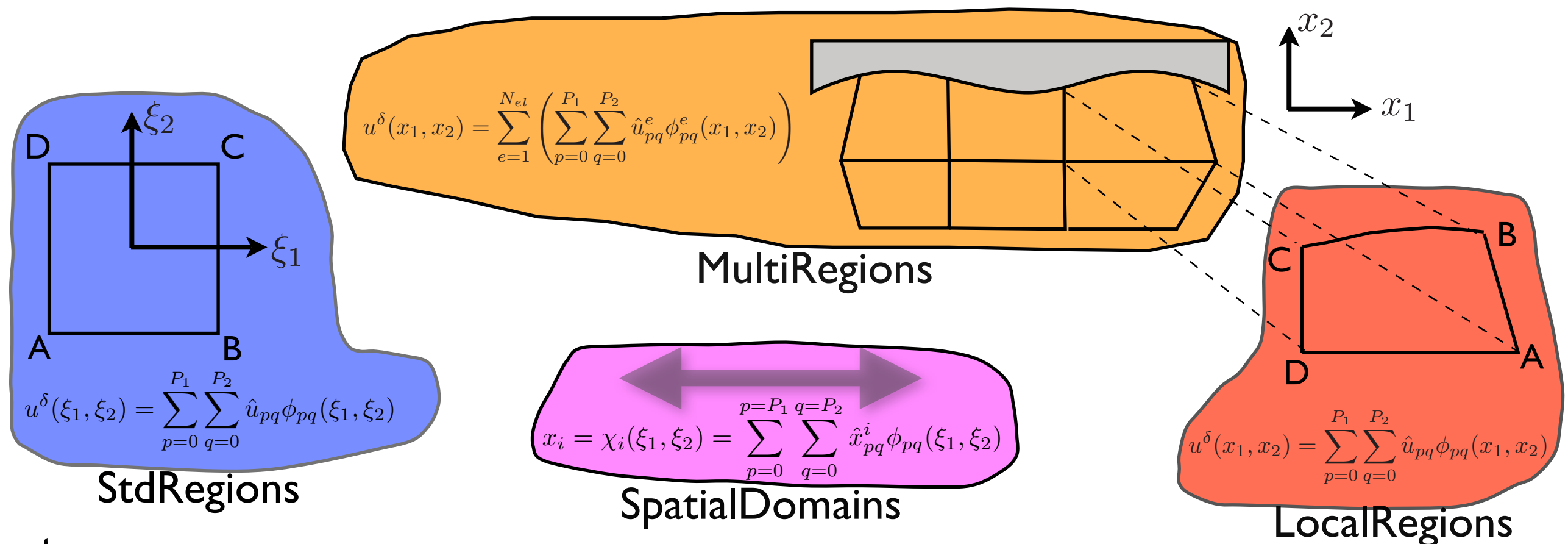
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Putting it all together

$$\sum_i \hat{v}_j \left\{ \sum_j \int_0^l \left[\frac{\partial \Phi_i^{\mathcal{H}}}{\partial x} \frac{\partial \Phi_j^{\mathcal{H}}}{\partial x} + \lambda \Phi_i^{\mathcal{H}} \Phi_j^{\mathcal{H}} \right] \hat{u}_j dx = \int \Phi_i^{\mathcal{H}} f^* dx \right\}$$

$$\mathbf{f}[i] = \int \Phi_i f^* dx = \sum_e^{nel} \sum_p \int_{\Omega^e} \phi_p(x) f^* dx = \sum_e^{nel} \sum_p \int_{\Omega^e} \phi_p(\chi^e(\xi)) f^* J^e d\xi$$

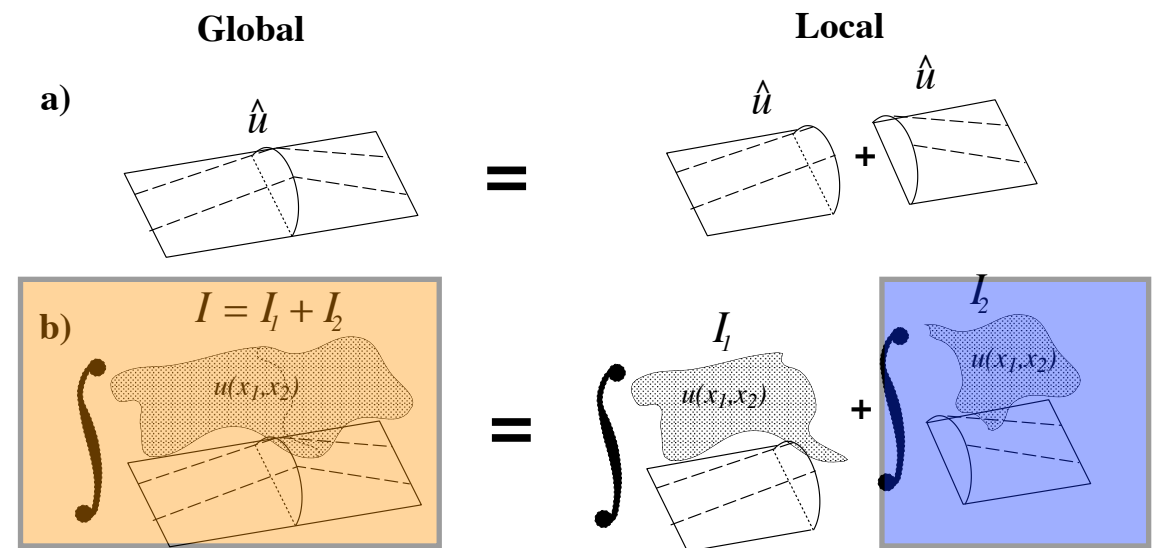
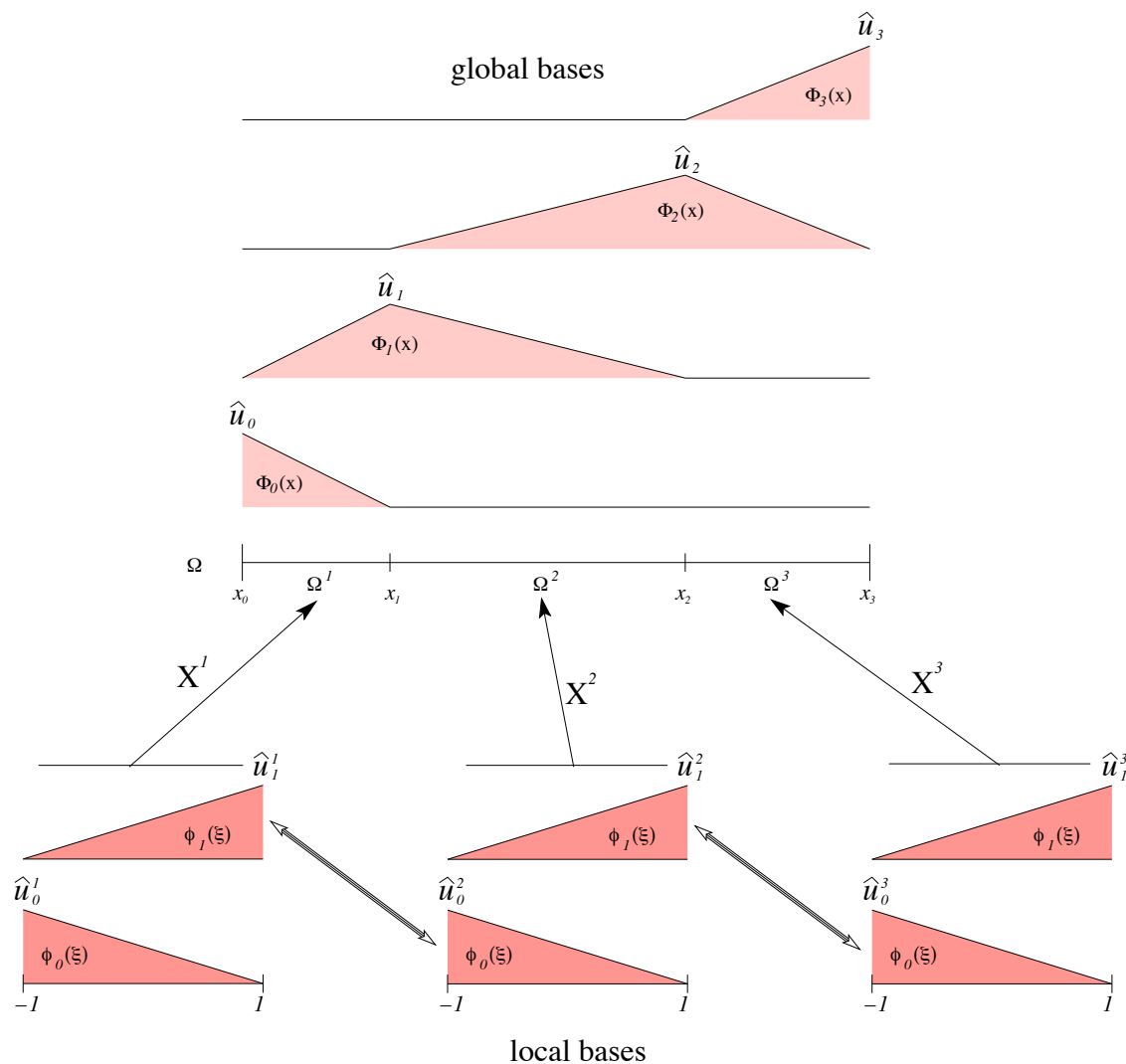


Outline

- 1D formulation of Helmholtz Problem
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1	Fundamental Concepts in One Dimension	1
1.1	Method of Weighted Residuals	2
1.2	Galerkin Formulation	5
1.2.1	Descriptive Formulation	5
1.2.2	Two-Domain Linear Finite Element Example	8
1.2.3	Mathematical Formulation	11
1.2.4	Mathematical Properties of the Galerkin Approximation	13
1.2.5	Residual Equation for C^0 Test and Trial Functions	15
1.3	One-Dimensional Expansion Bases	16
1.3.1	Elemental Decomposition: The h -Type Extension	16
1.3.2	Polynomial Expansions: The p -Type Extension	22
1.3.3	Modal Polynomial Expansions	28
1.3.4	Nodal Polynomial Expansions	31
1.4	Elemental Operations	34
1.4.1	Numerical Integration	34
1.4.2	Differentiation	39
1.5	Error Estimates	42
1.5.1	h -Convergence of Linear Finite Elements	42
1.5.2	L^2 Error of the p -Type Interpolation in a Single Element	44
1.5.3	General Error Estimates for hp Elements	45
1.6	Implementation of a 1D Spectral/ hp Element Solver:	46
1.6.1	Exercises	46
1.6.2	Convergence Examples	50
3	Multi-dimensional Formulation	98
3.1	Local Elemental Operations	99
3.1.1	Integration within the Standard Region Ω_{st}	99
3.1.2	Differentiation in the Standard Region Ω_{st}	104
3.1.3	Operations within General Shaped Elements	109
3.1.4	Discrete Evaluation of the Surface Jacobian	115
3.1.5	Elemental Projections and Transformations	118
3.1.6	Sum Factorisation/Tensor Product Operations	129
3.2	Global Operations	134
3.2.1	Global Assembly and Connectivity	135
3.2.2	Global Matrix System	146
3.2.3	Static Condensation/Substructuring	149
3.2.4	Global Boundary System Numbering and Ordering to Enforce Dirichlet Boundary Conditions	153
3.3	Pre- and Post-Processing Issues	157
3.3.1	Boundary Condition Discretisation	158
3.3.2	Elemental Boundary Transformation	158
3.3.3	Mesh Generation for Spectral/ hp Element Discretisation	161
3.3.4	Global Coarse Meshing	162
3.3.5	High-Order Mesh Generation	163
3.3.6	Particle Tracking in Spectral/ hp Element Discretisations	172
3.4	Exercises: Implementation of a 2D Spectral/ hp Element solver for a Global Projection Problem Using a C^0 Galerkin Formulation	181

Global Assembly



$$I^e[i] = \sum_e^{nel} \sum_p \int_{\Omega^e} \phi_p(\chi^e(\xi)) f^* J^e d\xi$$

$$f[i] = \int \Phi_i f^* dx$$

Do $e = 1, N_{el}$

Do $i = 0, N_{el}^e - 1$

$$f[\text{map}[e][i]] = f[\text{map}[e][i]] + I^e[i]$$

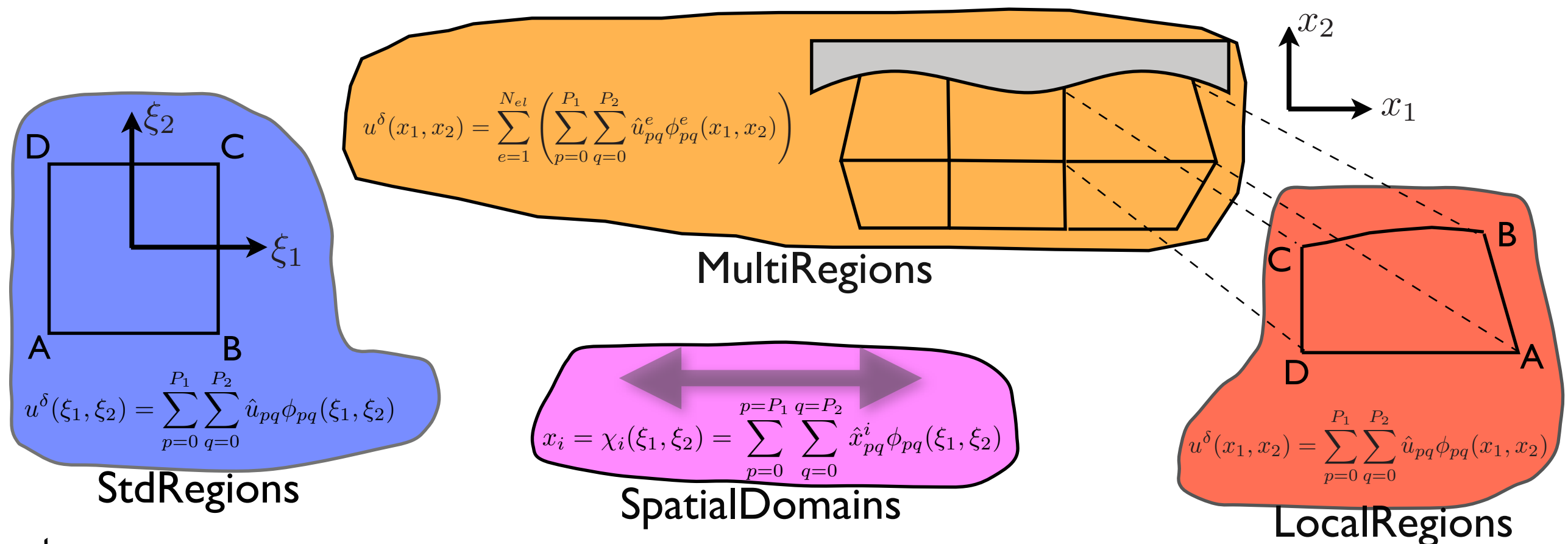
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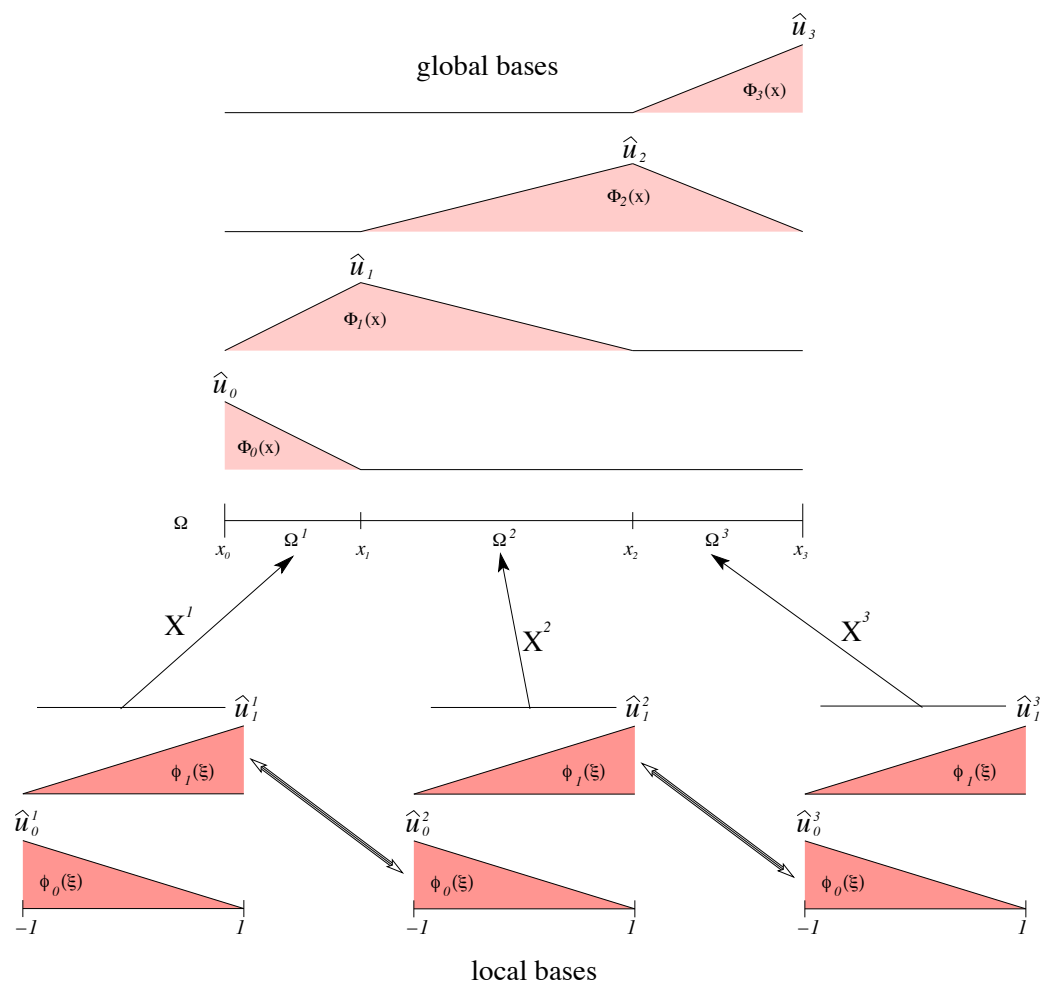
Matrix Construction

$$\sum_i \hat{v}_j \left\{ \sum_j \int_0^l \left[\frac{\partial \Phi_i^{\mathcal{H}}}{\partial x} \frac{\partial \Phi_j^{\mathcal{H}}}{\partial x} + \lambda \Phi_i^{\mathcal{H}} \Phi_j^{\mathcal{H}} \right] \hat{u}_j dx = \int \Phi_i^{\mathcal{H}} f^* dx \right\}$$

$$\begin{aligned} \mathbf{M}[i][j] &= \int_{\Omega} \Phi_i \Phi_j dx = \sum_e^{nel} \sum_p \sum_q \int_{\Omega^e} \phi_p(x) \phi_q(x) dx \\ &= \sum_e^{nel} \sum_p \sum_q \int_{-1}^1 \phi_p(\chi^e(\xi)) \phi_q(\chi^e(\xi)) j^e d\xi \end{aligned}$$



Matrix construction



$$\text{map}[1][i] = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \quad \text{map}[2][i] = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \quad \text{map}[3][i] = \begin{Bmatrix} 2 \\ 3 \end{Bmatrix}$$

$$M_G = \begin{matrix} \mathcal{A}^T & & \\ & M^1 & \\ & & M^2 & \\ & & & M^3 & \\ & & & & M^4 \end{matrix} \begin{matrix} & & & & A \\ & I & & & \\ & & I & & \\ & & & I & \\ & & & & I & \\ I & & & & & I \end{matrix} = \begin{matrix} & & & & M^1 & \\ & & & & & M^2 & \\ & & & & & & M^3 & \\ & & & & & & & M^4 \end{matrix}$$

$$M_G = \begin{matrix} \mathcal{A}^T & & \\ & M^1 & \\ & & M^2 & \\ & & & M^3 & \\ & & & & M^4 \end{matrix} \begin{matrix} & & & & A \\ & I & & & \\ & & I & & \\ & & & I & \\ & & & & I & \\ I & & & & & I \end{matrix} = \begin{matrix} & & & & M^1 & \\ & & & & & M^2 & \\ & & & & & & M^3 & \\ & & & & & & & M^4 \end{matrix}$$

$$\left. \begin{array}{l} \text{Do } e = 1, N_{el} \\ \quad \text{Do } i = 0, N_m^e - 1 \\ \quad \quad \hat{u}^e[i] = \hat{u}_g[\text{map}[e][i]] \\ \quad \text{continue} \\ \text{continue} \end{array} \right\} \Leftrightarrow \hat{u}_l = \mathcal{A} \hat{u}_g,$$

$$\left. \begin{array}{l} \text{Do } e = 1, N_{el} \\ \quad \text{Do } i = 0, N_m^e - 1 \\ \quad \quad \hat{u}_g[\text{map}[e][i]] = \hat{u}_g[\text{map}[e][i]] + \hat{u}^e[i] \\ \quad \text{continue} \\ \text{continue} \end{array} \right\} \Leftrightarrow \hat{u}_g = \mathcal{A}^T \hat{u}_l$$

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1	Fundamental Concepts in One Dimension	1
1.1	Method of Weighted Residuals	2
1.2	Galerkin Formulation	5
1.2.1	Descriptive Formulation	5
1.2.2	Two-Domain Linear Finite Element Example	8
1.2.3	Mathematical Formulation	11
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1.3.1	Elemental Decomposition: The h -Type Extension	16
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1.3.3	Modal Polynomial Expansions	28
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1.4	Elemental Operations	34
1.4.1	Numerical Integration	34
1.4.2	Differentiation	39
1.5	Error Estimates	42
1.5.1	h -Convergence of Linear Finite Elements	42
1.5.2	L^2 Error of the p -Type Interpolation in a Single Element	44
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1.6	Implementation of a 1D Spectral/ hp Element Solver:	46
1.6.1	Exercises	46
1.6.2	Convergence Examples	50
3	Multi-dimensional Formulation	98
3.1	Local Elemental Operations	99
3.1.1	Integration within the Standard Region Ω_{st}	99
3.1.2	Differentiation in the Standard Region Ω_{st}	104
3.1.3	Operations within General Shaped Elements	109
3.1.4	Discrete Evaluation of the Surface Jacobian	115
3.1.5	Elemental Projections and Transformations	118
3.1.6	Sum Factorisation/Tensor Product Operations	129
3.2	Global Operations	134
3.2.1	Global Assembly and Connectivity	135
3.2.2	Global Matrix System	146
3.2.3	Static Condensation/Substructuring	149
3.2.4	Global Boundary System Numbering and Ordering to Enforce Dirichlet Boundary Conditions	153
3.3	Pre- and Post-Processing Issues	157
3.3.1	Boundary Condition Discretisation	158
3.3.2	Elemental Boundary Transformation	158
3.3.3	Mesh Generation for Spectral/ hp Element Discretisation	161
3.3.4	Global Coarse Meshing	162
3.3.5	High-Order Mesh Generation	163
3.3.6	Particle Tracking in Spectral/ hp Element Discretisations	172
3.4	Exercises: Implementation of a 2D Spectral/ hp Element solver for a Global Projection Problem Using a C^0 Galerkin Formulation	181

2/3D Helmholtz problem

$$\nabla^2 u - \lambda u = -f$$

Integral
formulation:

$$\int_{\Omega} v \nabla^2 u \, d\mathbf{x} - \int_{\Omega} \lambda u \, d\mathbf{x} = - \int_{\Omega} v f \, d\mathbf{x}$$

Divergence
theorem:

$$\int_{\Omega} v \nabla^2 u \, d\mathbf{x} = \oint v \frac{\partial u}{\partial n} \, ds - \int_{\Omega} \nabla v \nabla u \, d\mathbf{x}$$

Weak form:

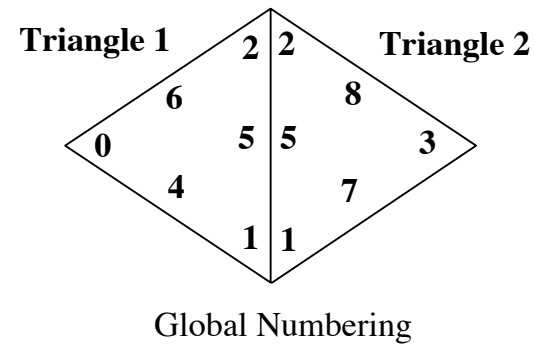
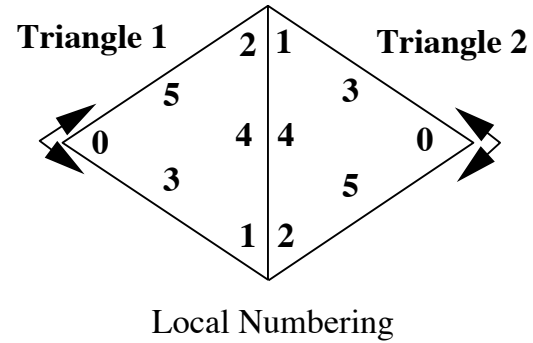
$$\int_{\Omega} \nabla v \nabla u + \lambda u \, d\mathbf{x} = \int_{\Omega} v f \, d\mathbf{x} + \oint_{\partial\Omega} v \frac{\partial u}{\partial n} \, ds$$

Dirichlet BC: $u = u^{\mathcal{D}} + u^{\mathcal{H}}$

Neumann BC: $\oint_{\partial\Omega} v g_{\mathcal{N}} \, ds$



Global assembly



$$\hat{\mathbf{u}}_l = \begin{bmatrix} \hat{\mathbf{u}}^1[0] \\ \hat{\mathbf{u}}^1[1] \\ \hat{\mathbf{u}}^1[2] \\ \hat{\mathbf{u}}^1[3] \\ \hat{\mathbf{u}}^1[4] \\ \hat{\mathbf{u}}^1[5] \\ \dots \\ \hat{\mathbf{u}}^2[0] \\ \hat{\mathbf{u}}^2[1] \\ \hat{\mathbf{u}}^2[2] \\ \hat{\mathbf{u}}^2[3] \\ \hat{\mathbf{u}}^2[4] \\ \hat{\mathbf{u}}^2[5] \end{bmatrix} = \begin{bmatrix} 1 & & & & & & & & & & & & & & & & \\ & 1 & & & & & & & & & & & & & & & \\ & & 1 & & & & & & & & & & & & & & \\ & & & 1 & & & & & & & & & & & & & \\ & & & & 1 & & & & & & & & & & & & \\ & & & & & 1 & & & & & & & & & & & \\ & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ & & & & & & 1 & & & & & & & & & & \\ & & & & & & & 1 & & & & & & & & & \\ & & & & & & & & 1 & & & & & & & & \\ & & & & & & & & & 1 & & & & & & & \\ & & & & & & & & & & 1 & & & & & & \\ & & & & & & & & & & & 1 & & & & & \end{bmatrix} \begin{bmatrix} \hat{\mathbf{u}}_g[0] \\ \hat{\mathbf{u}}_g[1] \\ \hat{\mathbf{u}}_g[2] \\ \hat{\mathbf{u}}_g[3] \\ \hat{\mathbf{u}}_g[4] \\ \hat{\mathbf{u}}_g[5] \\ \hat{\mathbf{u}}_g[6] \\ \hat{\mathbf{u}}_g[7] \\ \hat{\mathbf{u}}_g[8] \end{bmatrix}.$$

$$\text{map}[1][i] = \begin{cases} 0 \\ 1 \\ 2 \\ 4 \\ 5 \\ 6 \end{cases}$$

$$\text{map}[2][i] = \begin{cases} 3 \\ 2 \\ 1 \\ 8 \\ 5 \\ 7 \end{cases}.$$

$$\left. \begin{array}{l} \text{Do } e = 1, N_{el} \\ \quad \text{Do } i = 0, N_m^e - 1 \\ \quad \quad \hat{\mathbf{u}}^e[i] = \text{sign}[e][i] \cdot \hat{\mathbf{u}}_g[\text{map}[e][i]] \\ \quad \text{continue} \\ \text{continue} \end{array} \right\} \Leftrightarrow \hat{\mathbf{u}}_l = \mathcal{A} \hat{\mathbf{u}}$$

$$\left. \begin{array}{l} \text{Do } e = 1, N_{el} \\ \quad \text{Do } i = 0, N_m^e - 1 \\ \quad \quad \hat{\mathbf{I}}_g[\text{map}[e][i]] = \hat{\mathbf{I}}_g[\text{map}[e][i]] \\ \quad \quad \quad + \text{sign}[e][i] \cdot \hat{\mathbf{I}}^e[i] \\ \quad \text{continue} \\ \text{continue} \end{array} \right\} \Leftrightarrow \hat{\mathbf{I}}_g = \mathcal{A}^T \hat{\mathbf{I}}_l.$$

2D Global Assembly: Modal

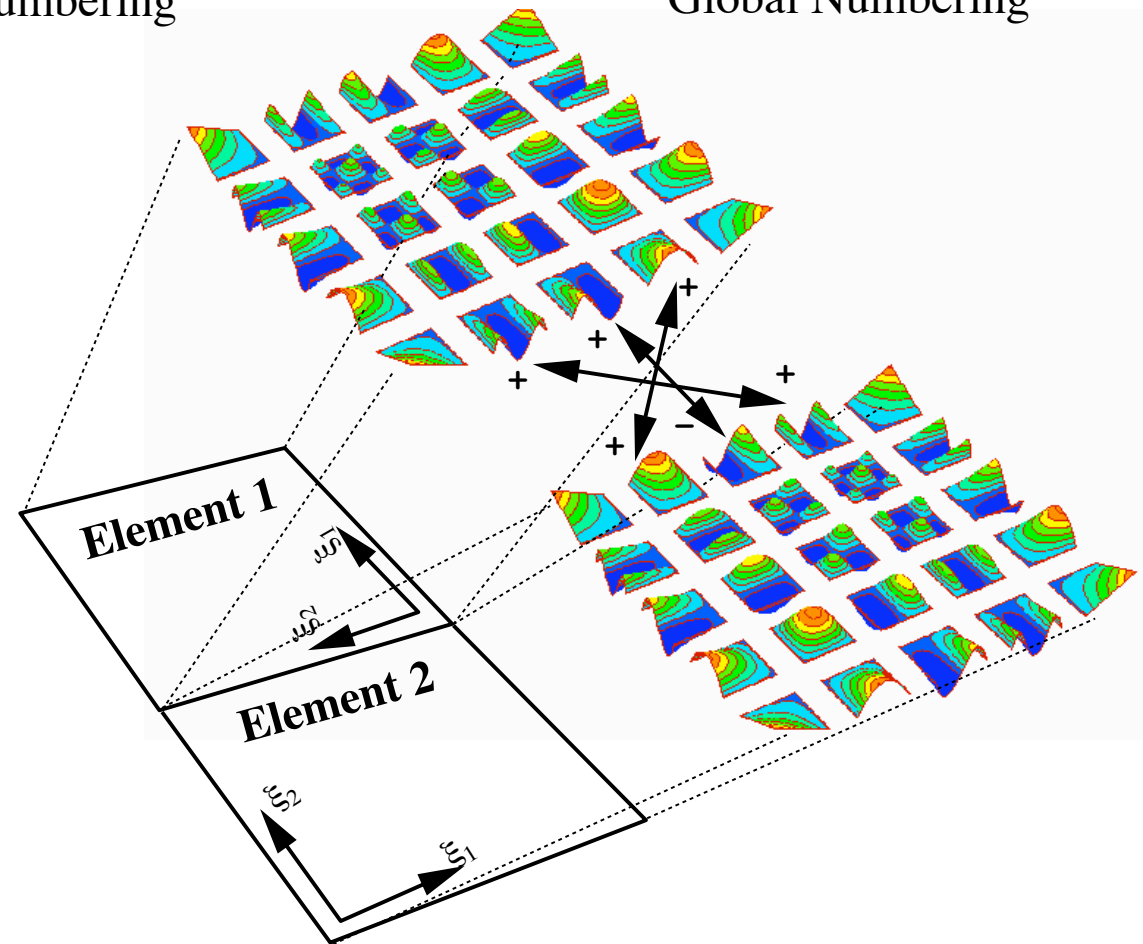
Element 1			Element 2		
1	4 5 6	0	2	7 8 9	1
7 8 9		15 14 13	10 11 12		6 5 4
	12 11 10			15 14 13	
2		3	3		4

Element 1			Element 2		
1	6 7 8	0	0	21 22 23	5
9 10 11		17 16 15	15 16 17		20 19 18
	14 13 12			26 25 24	
2		3	3		4

Local Numbering

Global Numbering

Sign change
required when
local coordinates
reversed



2D Global Assembly: Nodal

Element 1

Element 2

1	6	5	4	0	2	9	8	7	1
7				15	10				6
8				14	11				5
9				13	12				4
2	10	11	12	3	3	13	14	15	0

Element 1

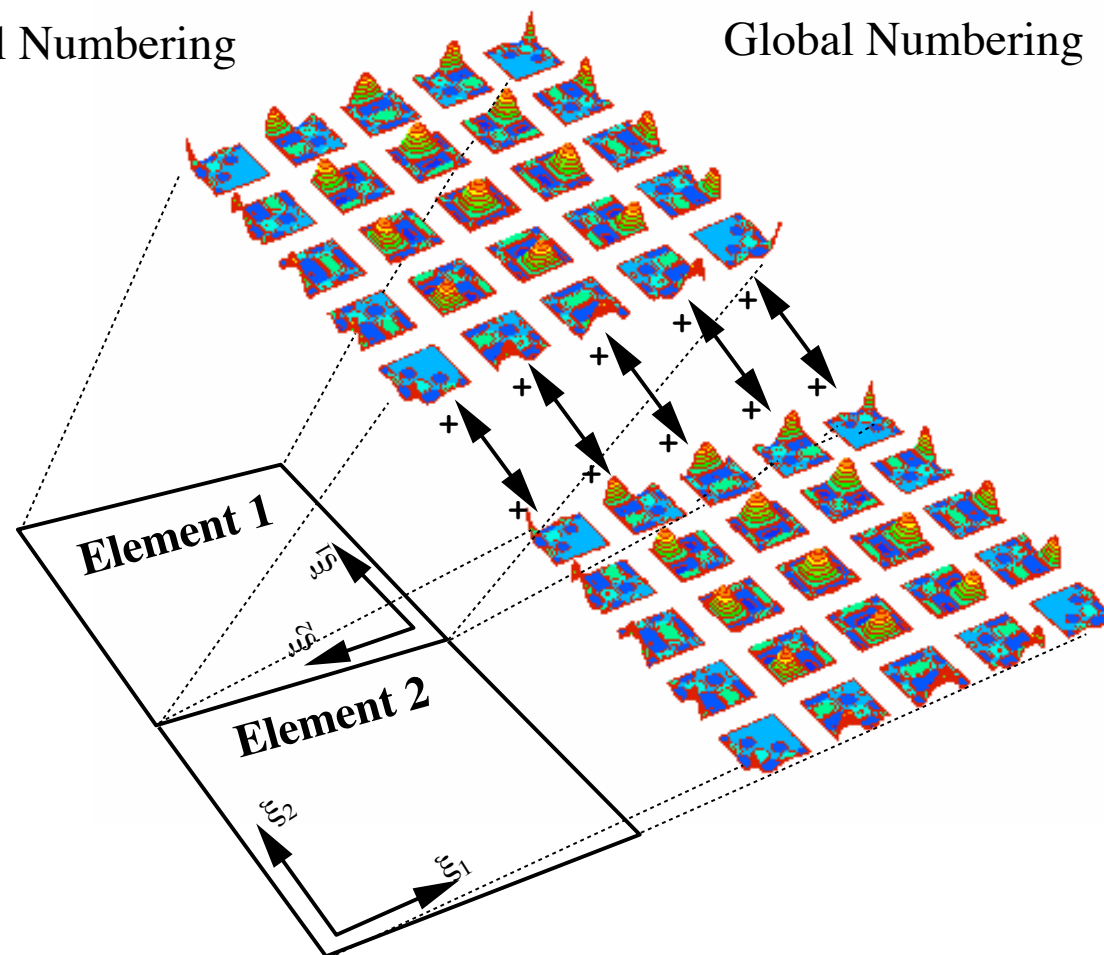
Element 2

1	8	7	6	0	0	23	22	21	5
9				17	17				20
10				16	16				19
11				15	15				18
2	12	13	14	3	3	24	25	26	4

Local Numbering

Global Numbering

Numbering
altered when
local coordinates
reversed



Nektar++ code

