Visualization of High-Order Finite Element Methods

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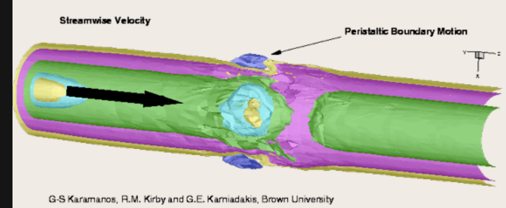
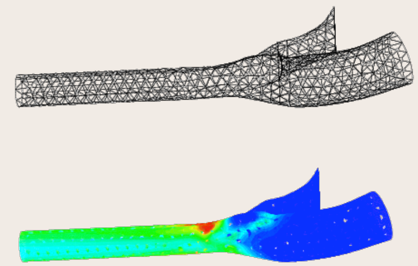
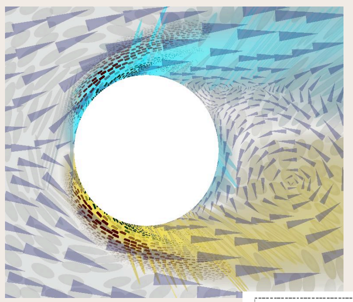
Collage of Applications

Aerodynamics

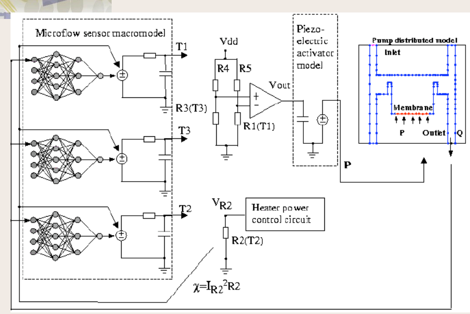


Biomechanics

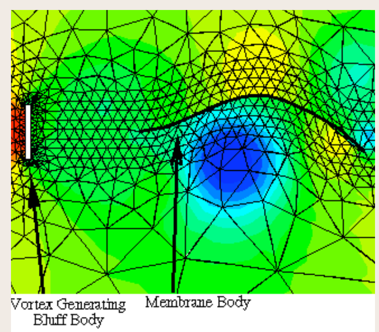




Scientific Visualization



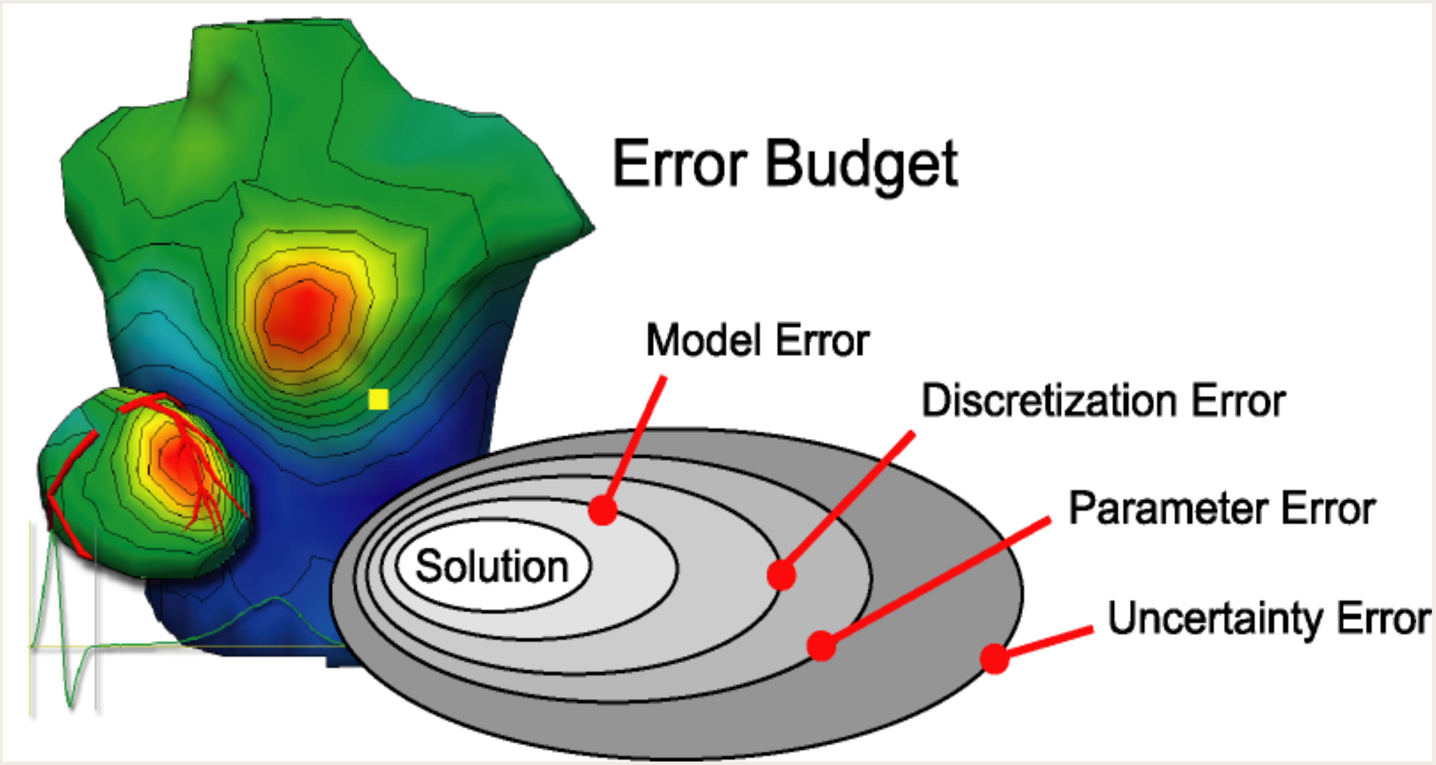
MEMS



Fluid-Structure

Visualization is a means of exploration and explanation

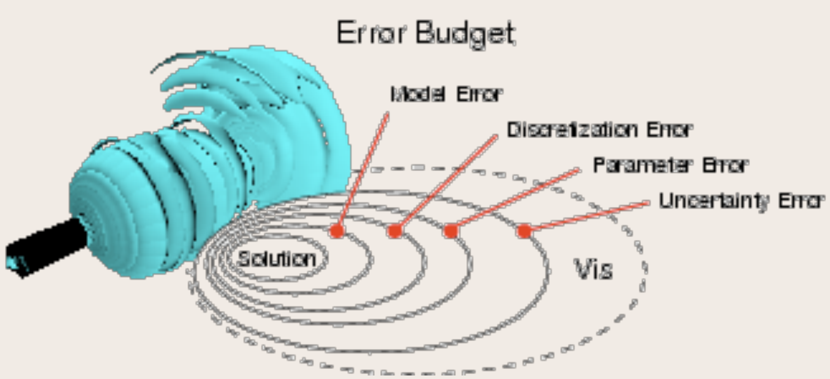
Simulation Science Error Budget



What About Visualization Error?

Visualization of High-Order Solutions: Motivation

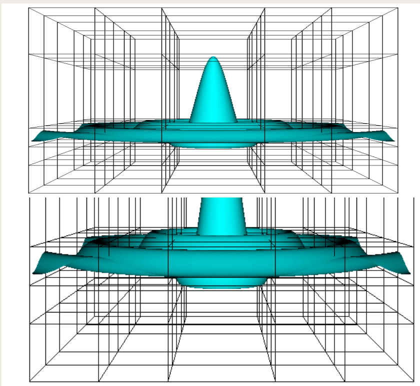
- Visualization introduces error
- This error can be minimized if we take into account our knowledge of the data



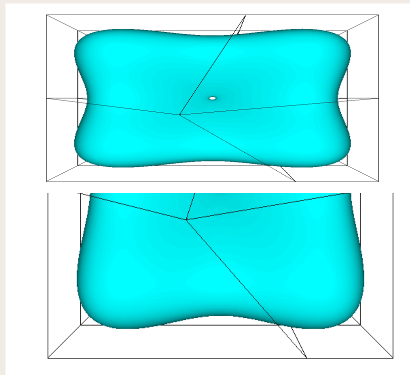
Visualization of High-Order Method Results Using Ray -Tracing

Blake Nelson and Robert M. Kirby, ``Ray-Tracing Polymorphic Multi-Domain Spectral/*hp* Elements for Isosurface Rendering'', *IEEE Transactions on Visualization and Computer Graphics*, Vol. 12, Number 1, pages 114-125, 2006.

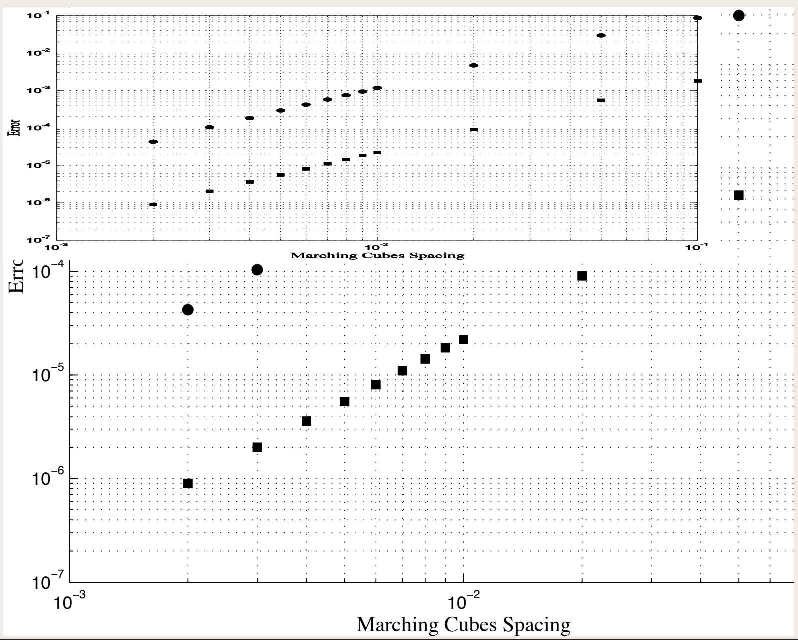
What is Typically Done?



Example 1 (circles)



Example 2 (squares)



Marching Cubes Approach

• Sample the data (i.e. evaluate high-order finite elements) on a lattice

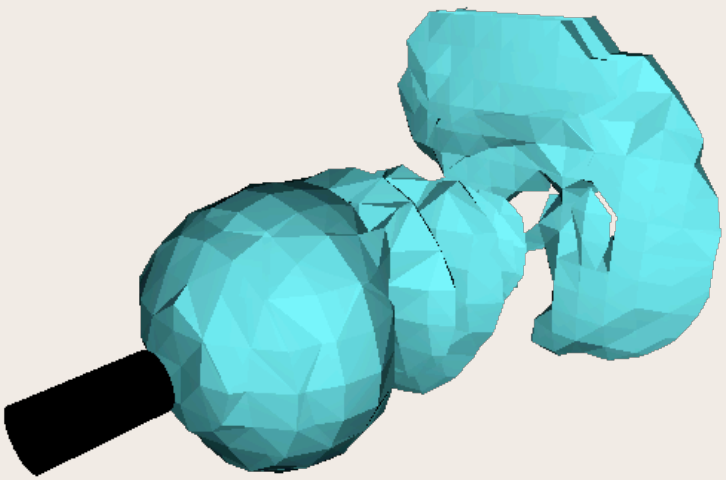
• On each edge of the lattice, assume a linear interpolant; find roots

• Based upon roots found on edges, generate triangles which approximate the isosurface

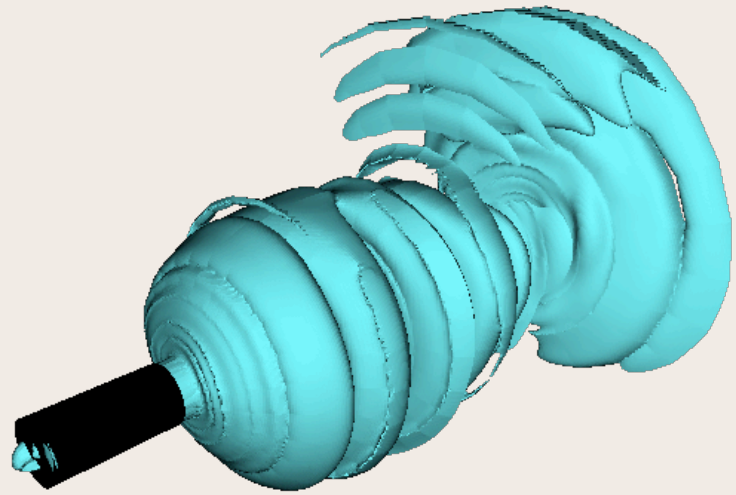
• Provide triangles to the rendering pipeline

Low-Order Approximation to One's High-Order Finite Element Data

Marching Cubes Visualization



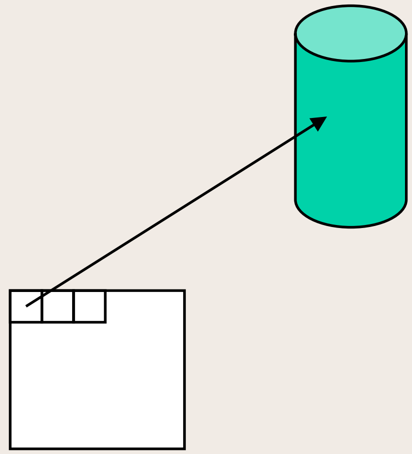
Coarse Marching
Cubes Refinement



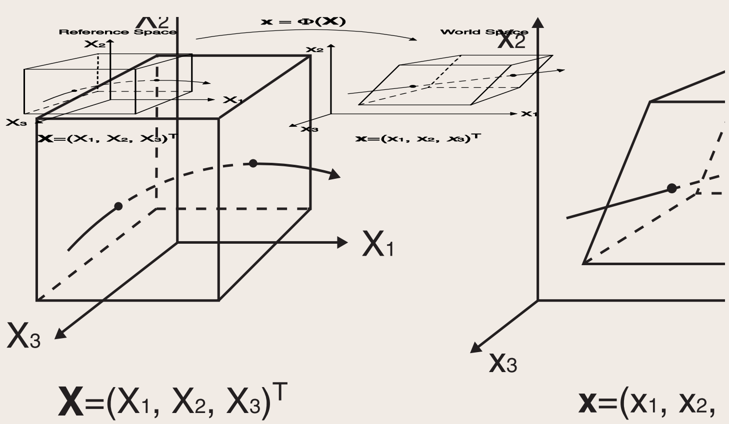
Highly-Refined Marching
Cubes Refinement
(2,921,875 Voxels)

Refinement is certainly a strategy, but is it the only strategy?

Ray Tracing Basics



- For each pixel, cast a ray from the eye-point through the pixel into the domain
- Determine elements through which the ray traverses
- Find the first intersection between the isosurface of interest and the ray

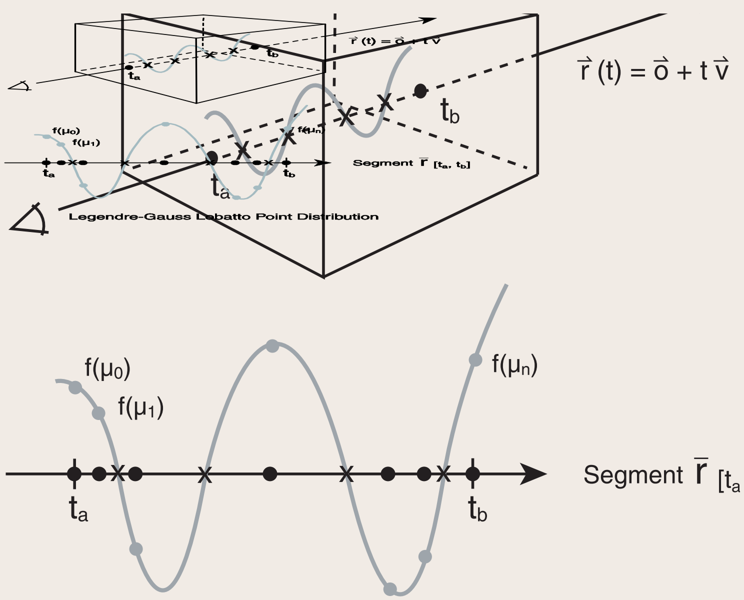


<u>Challenges with High-Order Finite Elements</u>

- Curved Boundaries
- Elemental Mappings
- Efficient and Robust Root-Finding

Ray Tracing For High-Order Elements

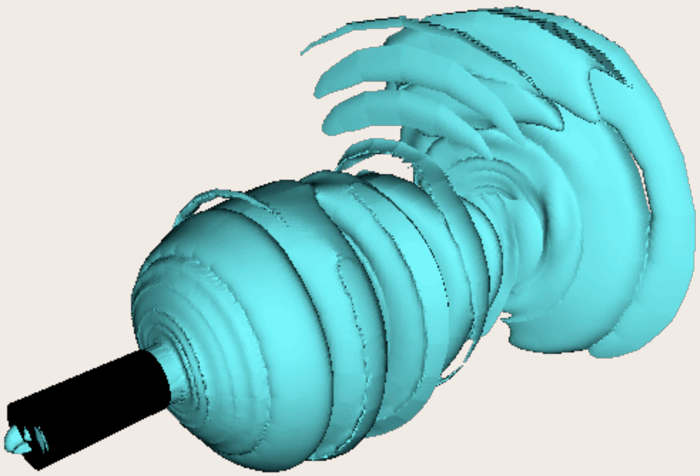
$$p^{e}(\vec{x}) = \sum_{k} \hat{p}_{k}^{e} \varphi(\vec{x})$$



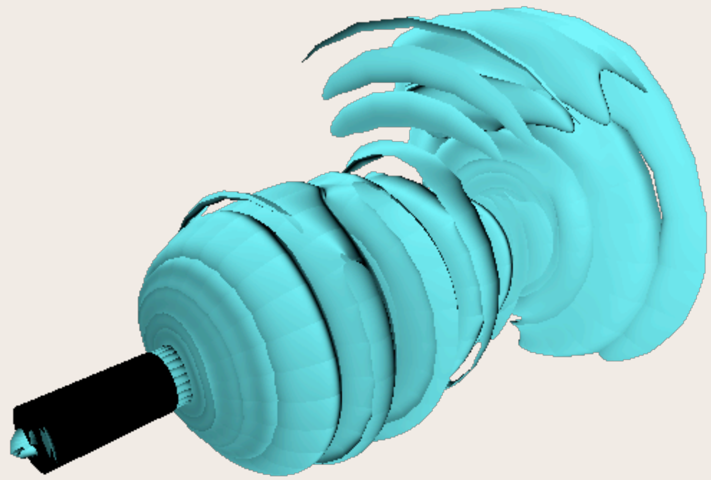
Legendre-Gauss Lobatto Point Distribution

- For each pixel, cast a ray from the eye-point through the pixel into the domain
- Determine elements through which the ray traverses
- Sample high-order FE solution at quadrature points along the segment
- Project high-order FE solution to Legendre basis along the ray (with error control)
- Use QR to find all the roots

Ray Tracing For High-Order Elements

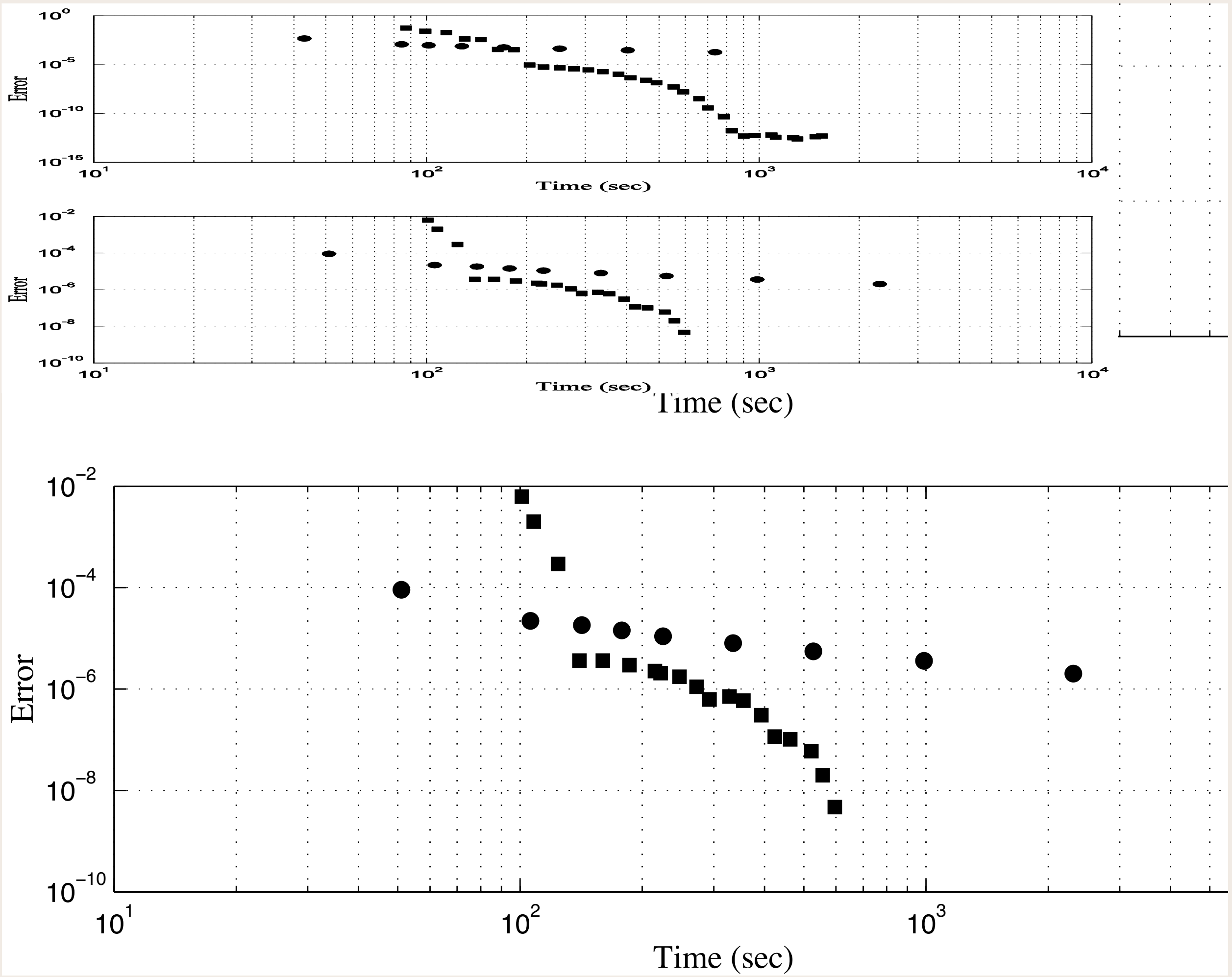


2,921,875 Voxels
Marching Cubes



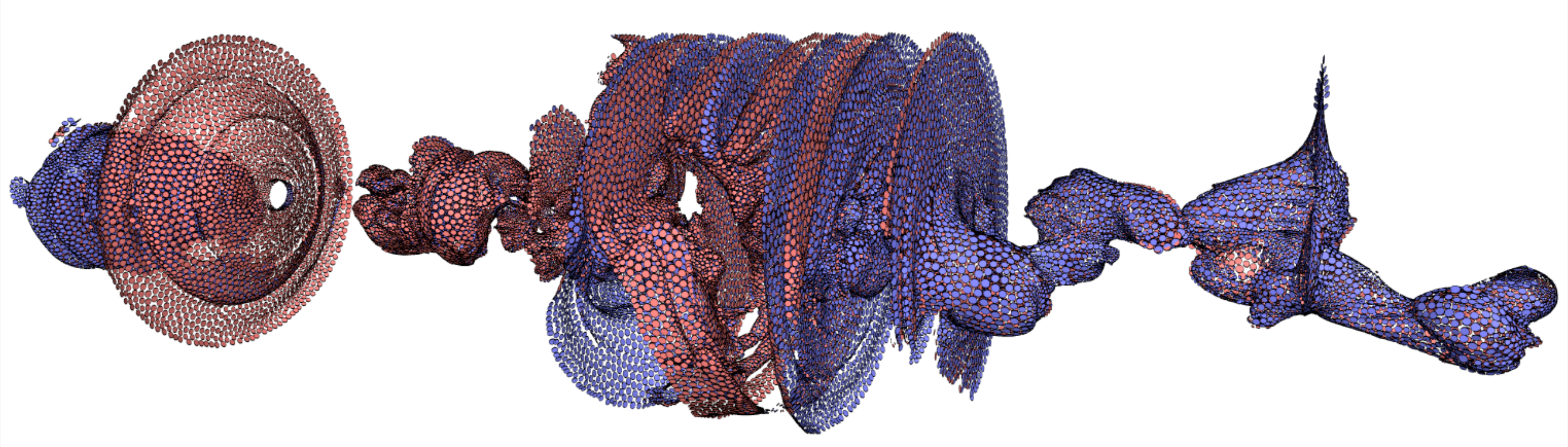
5,736 Elements
13-order Approximation

Error Versus Time For Two Example Discretizations



Circles: Marching Cubes
Squares: Ray-Tracing

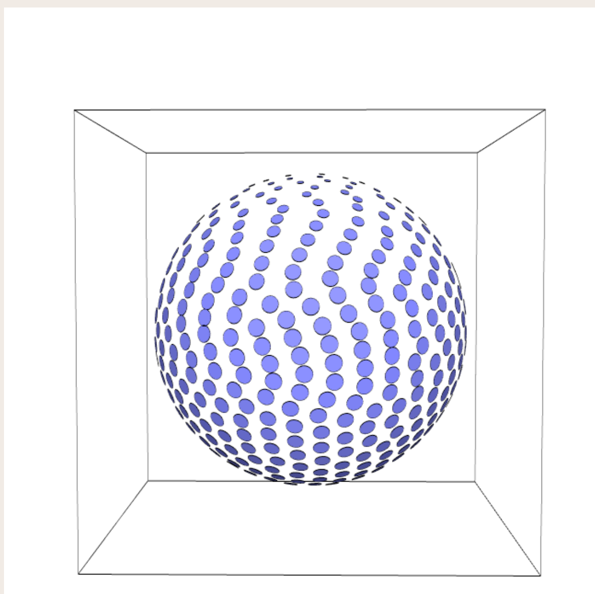
Visualization of High-Order Method Results Using Particle Methods



Miriah Meyer, Blake Nelson, Robert M. Kirby and Ross Whitaker, “Particle Systems for Efficient and Accurate Finite Element Visualization", *IEEE Transactions on Visualization and Computer Graphics*, In Press, 2007.

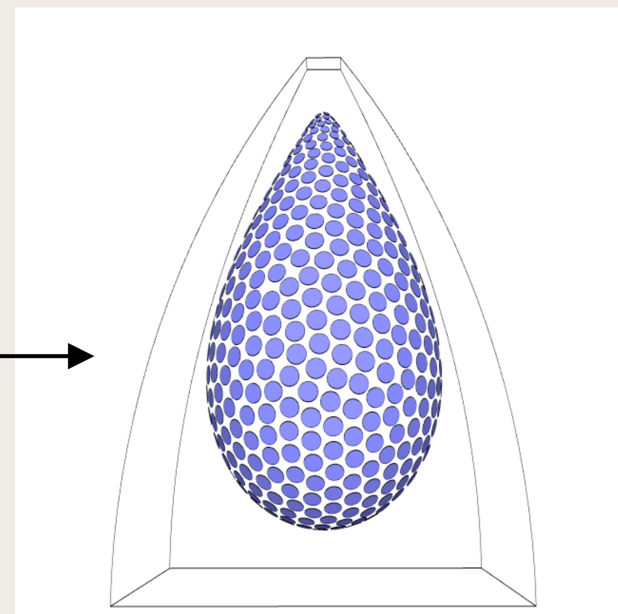
Particle System Approach

- Reference space particles
- Distributions computed from world space positions
- Adaptivity determined from world space Hessian



Reference Space

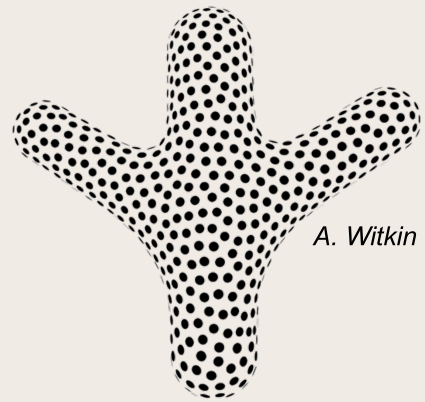
$T(\boldsymbol{u}) = \boldsymbol{x}$



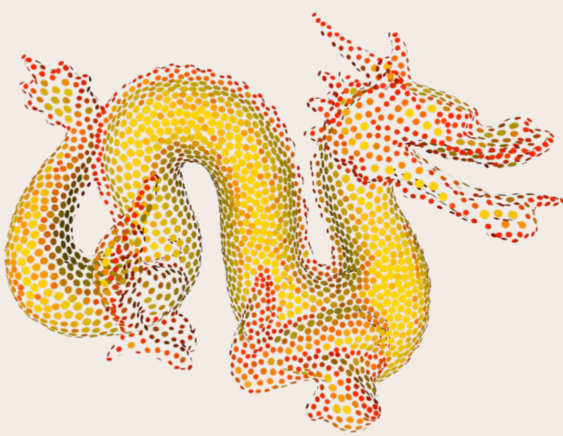
World Space

Particle System Basics

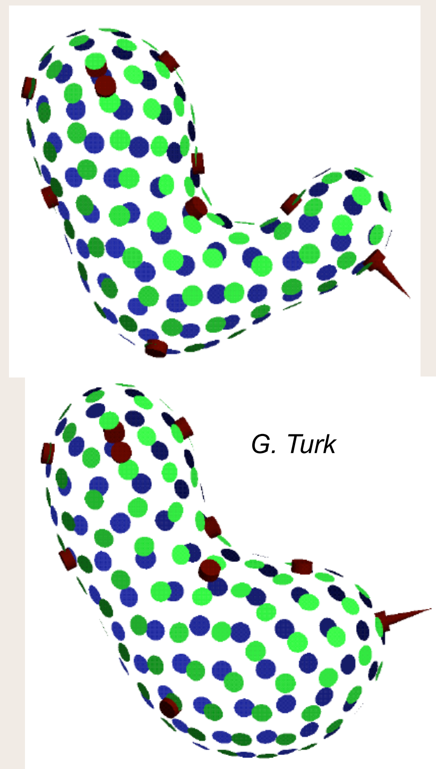
- For sampling surfaces
  - Witkin and Heckbert, 1992
  - Modeling
  - Visualization



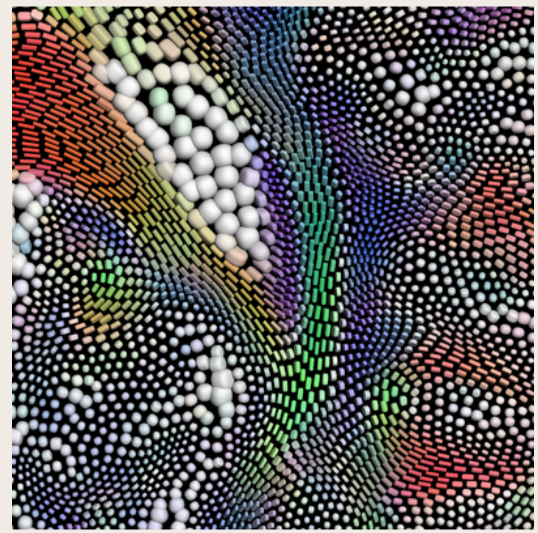
A. Witkin



M. Meyer



G. Turk



G. Kindlmann

Particle System Basics

1. Project onto surface

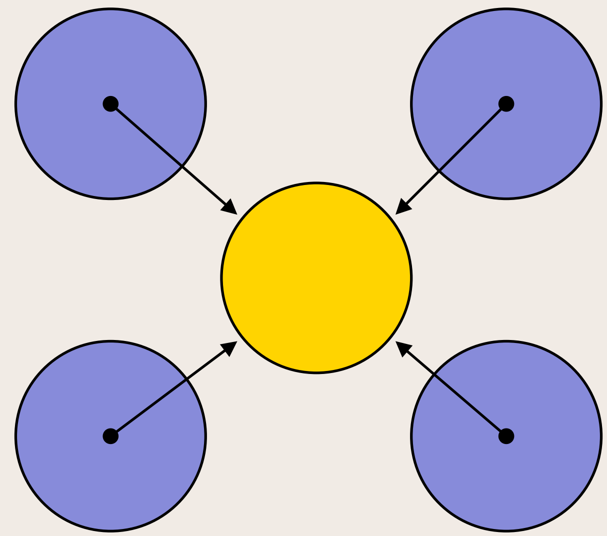
$$\mathbf{x}_i \leftarrow \mathbf{x}_i - F(\mathbf{x}_i)\frac{F_{\mathbf{x}}(\mathbf{x}_i)}{F_{\mathbf{x}}^{\mathrm{T}}(\mathbf{x}_i)F_{\mathbf{x}}(\mathbf{x}_i)}$$

2. Inter-particle forces

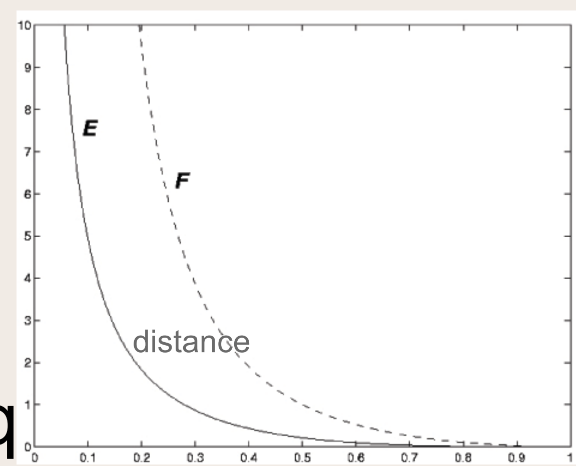
$$E_i = \sum_{j=1, j\neq i}^{m} E_{ij} = \sum_{j=1, j\neq i}^{m} E\left(\frac{|\mathbf{r}_{ij}|}{\sigma}\right)$$

$$\mathbf{v}_i = -\frac{\partial E_i}{\partial \mathbf{x}_i} = -\sum_{j=1, j\neq i}^{m} \frac{\partial E_{ij}}{\partial |\mathbf{r}_{ij}|}\frac{\mathbf{r}_{ij}}{|\mathbf{r}_{ij}|}$$

$$\mathbf{x}_i \leftarrow \mathbf{x}_i + \left(I - \frac{F_{\mathbf{x}}(\mathbf{x}_i)F_{\mathbf{x}}^{\mathrm{T}}(\mathbf{x}_i)}{F_{\mathbf{x}}^{\mathrm{T}}(\mathbf{x}_i)F_{\mathbf{x}}(\mathbf{x}_i)}\right)\mathbf{v}_i$$



3. Steady state when forces are eq



Particle System Basics

Adaptive distributions

$$\mathbf{r}_{ij} = \alpha_{ij}\left(\mathbf{x}_i - \mathbf{x}_j\right) = -\mathbf{r}_{ji}$$

$$\alpha_{ij} = \alpha_{ji} = \frac{1 + \rho D_{ij}\left(\frac{s}{2\pi}\right)}{s\beta}$$

ρ: angular density

s: planar seperation

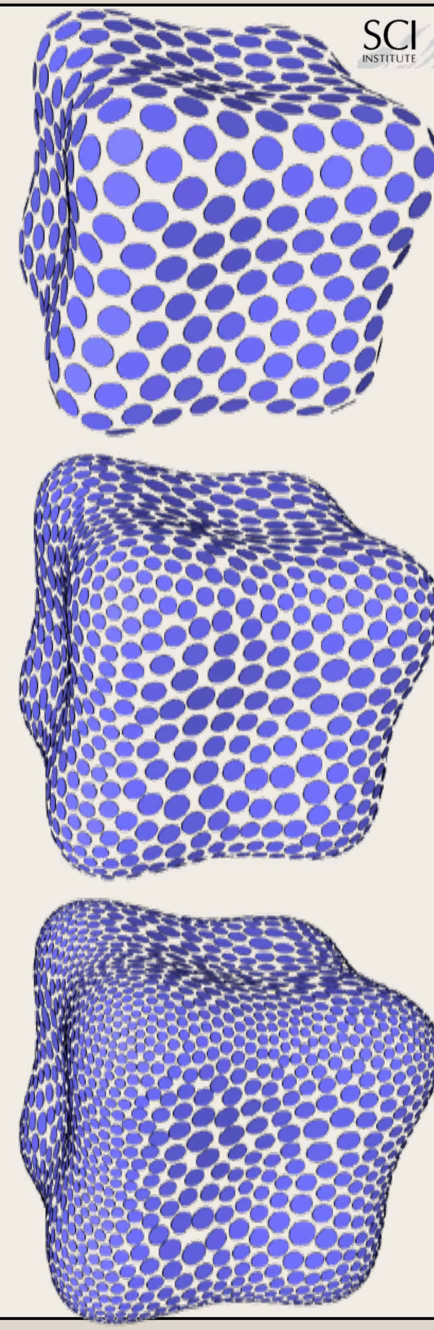
D: curvature magnitude

β: constant

$\rho = 0$

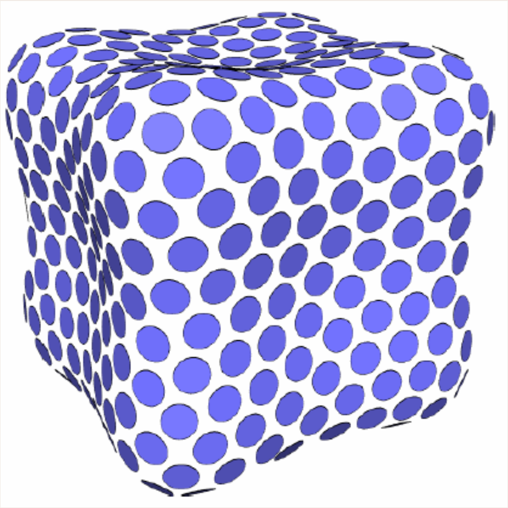
$\rho = 7$

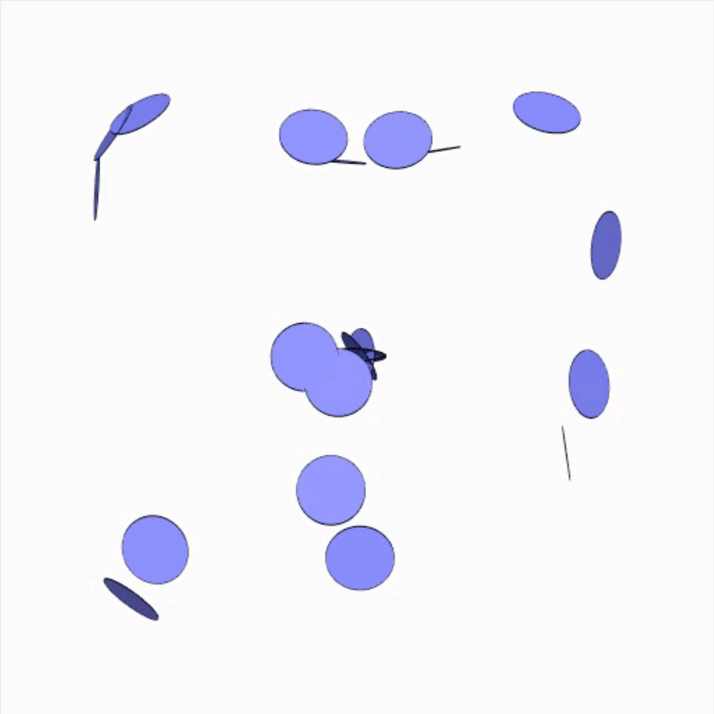
$\rho = 15$



Particle System Basics

- Population control
  - Split and delete particles
  - Based on energy of a hexagonal packing



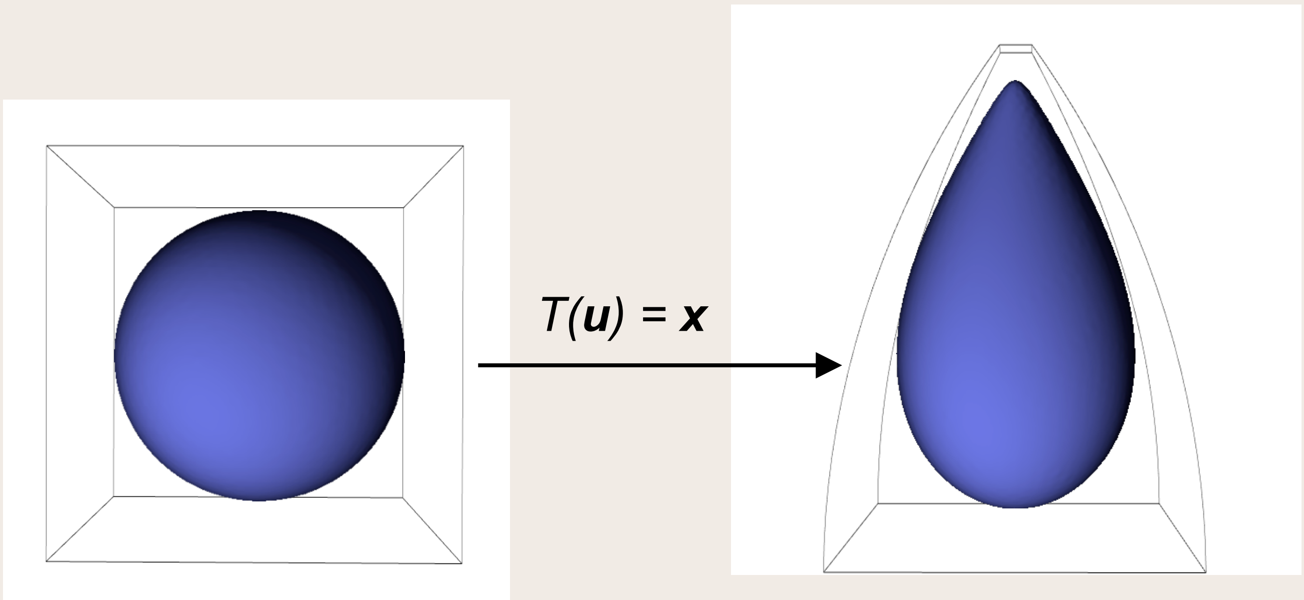


Isosurface Properties

- Gradient $F^i = \frac{\partial F}{\partial x^i} = \frac{\partial F^*}{\partial u_j} \frac{\partial u_j}{\partial x^i} = F_j^* K_j^i$

$$\boxed{K = J^{-1}}$$

- Hessian $\mathrm{F}^{\mathrm{ij}} = \frac{\partial}{\partial \mathrm{x}^{\mathrm{i}}}\left(\frac{\partial \mathrm{F}}{\partial \mathrm{x}^{\mathrm{j}}}\right) = \frac{\partial}{\partial \mathrm{u}_{\mathrm{k}}}\left(\frac{\partial \mathrm{F}}{\partial \mathrm{x}^{\mathrm{j}}} \frac{\partial \mathrm{u}_{\mathrm{k}}}{\partial \mathrm{x}^{\mathrm{i}}}\right) = \mathrm{F}_{\mathrm{lk}}^{*} \mathrm{K}_{\mathrm{l}}^{\mathrm{j}} \mathrm{K}_{\mathrm{k}}^{\mathrm{i}} - \mathrm{F}_{\mathrm{l}}^{*} \mathrm{K}_{\mathrm{l}}^{\mathrm{m}} \mathrm{J}_{\mathrm{kn}}^{\mathrm{m}} \mathrm{K}_{\mathrm{n}}^{\mathrm{j}} \mathrm{K}_{\mathrm{k}}^{\mathrm{i}}$



Reference Space

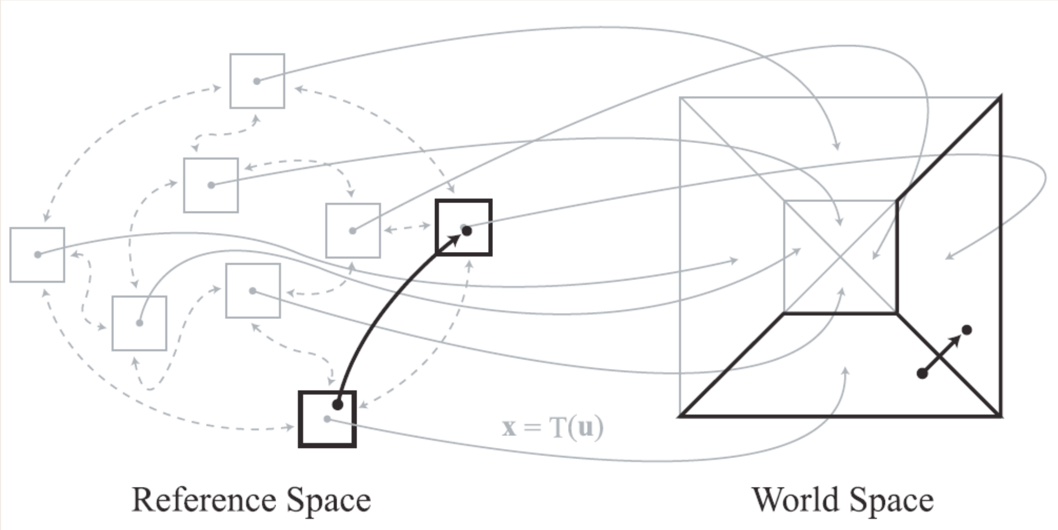
World Space

Reference Space Particles

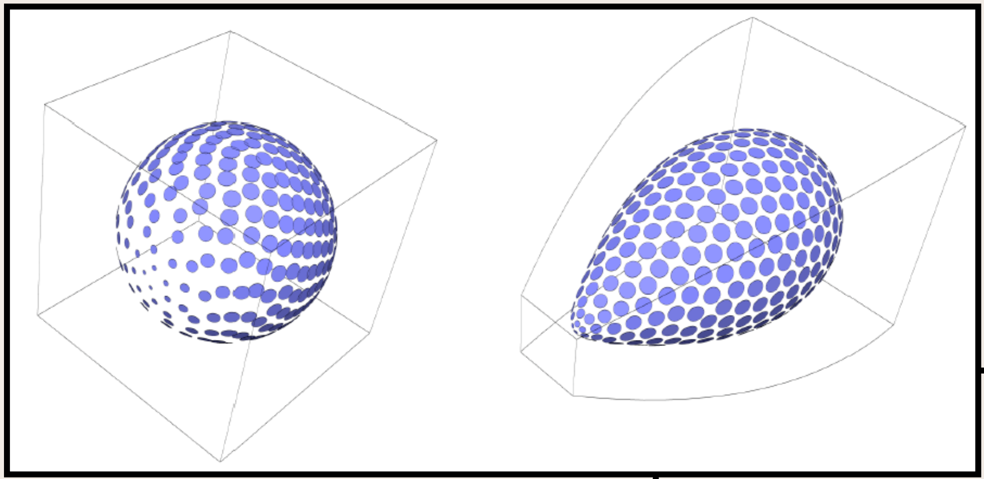
- Particles live in reference space
- Use world space positions to compute forces

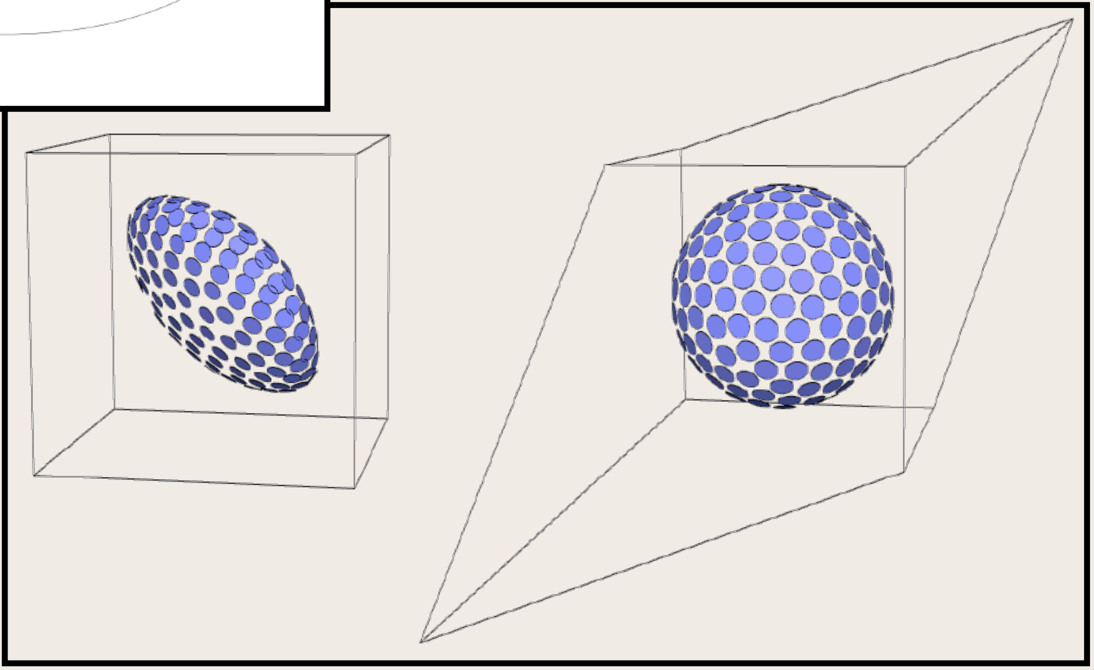
$$\mathbf{v}^* = K\mathbf{v}$$

- Move particle to a new element



Reference Space Particles

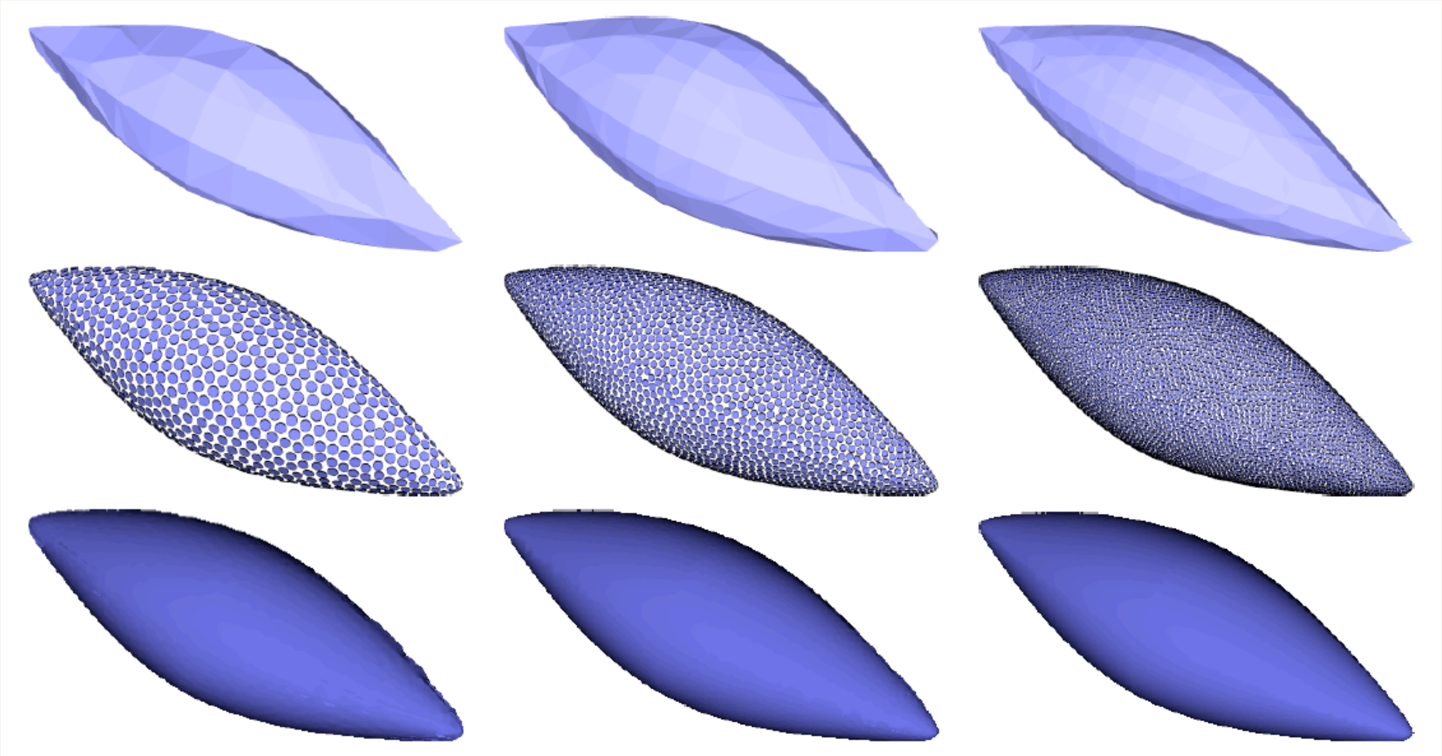




Results

Particle System vs. Marching Cubes

~20 seconds ~35 seconds ~70 seconds



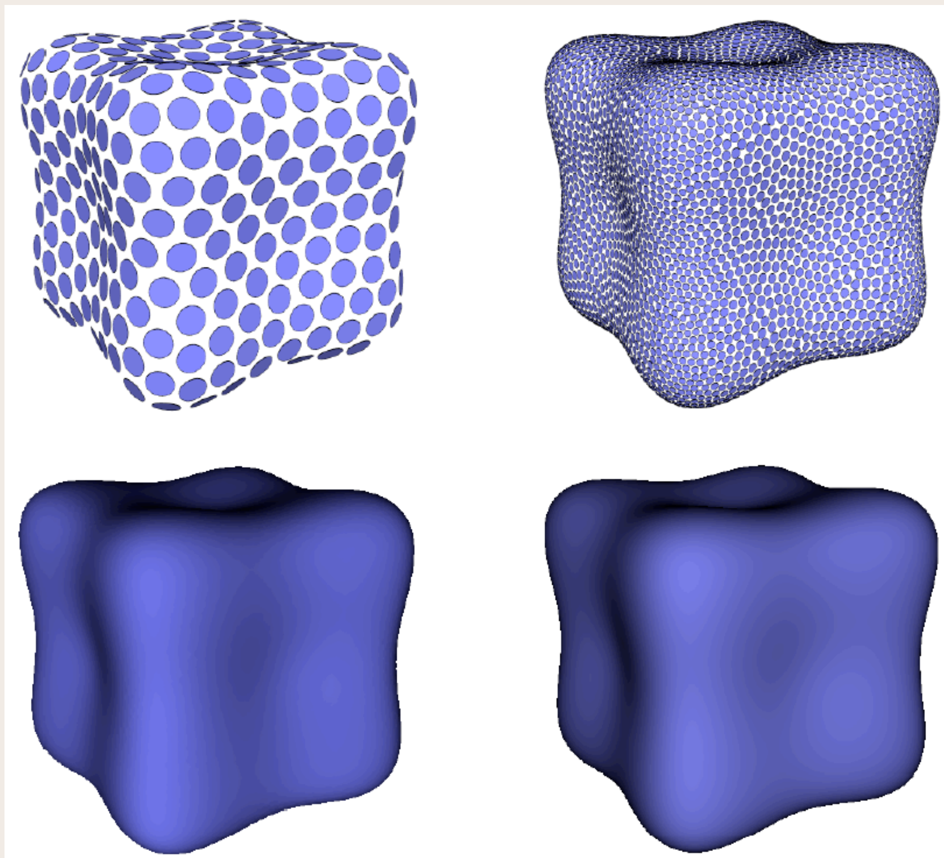
1300 particles *3200 particles* *8600 particles*

Results

Particle System vs. Raytracer

4 seconds,

500 particles



3 minutes,

6800 particles

GPU-splat rendering,

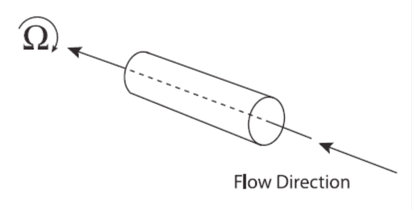
6800 particles

Raytraced

512x512 image,

6 minutes

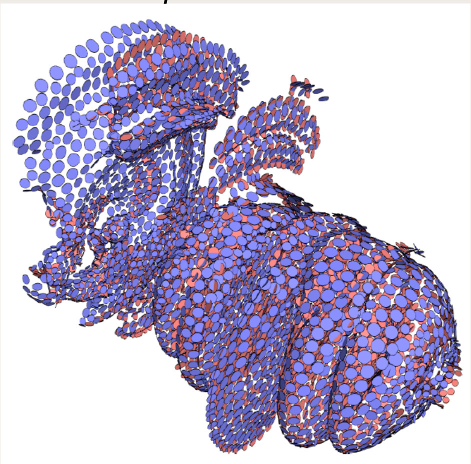
Results



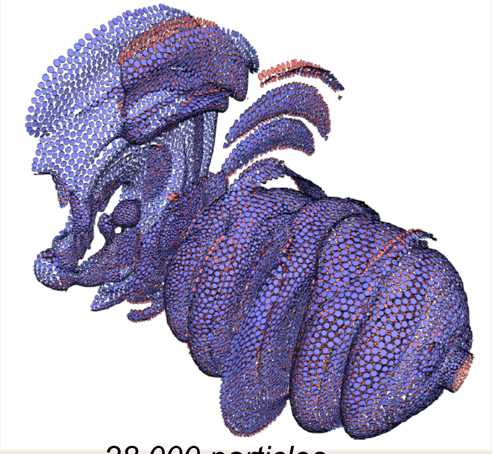
5736 elements

3rd order polynomials

5000 particles

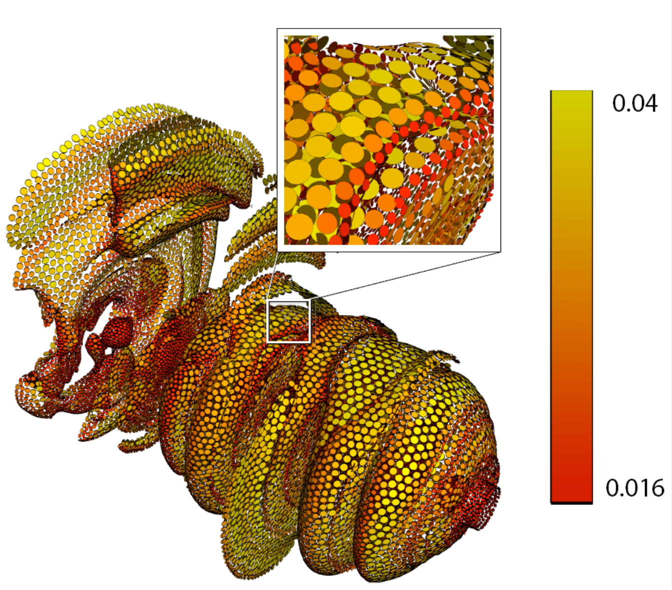


13,000 particles



28,000 particles

59,000 particles



Summary

- Visualization is part of the Simulation Science Pipeline

Exploration and Explanation

- Many Visualization Tools Use Low-Order Approximations To Render High-Order Finite Element Data

Goal: Exploit Structure of High-Order Finite Elements Directly

  – Ray-Tracing for High-Order Finite Elements

  – Particle Methods for High-Order Finite Element