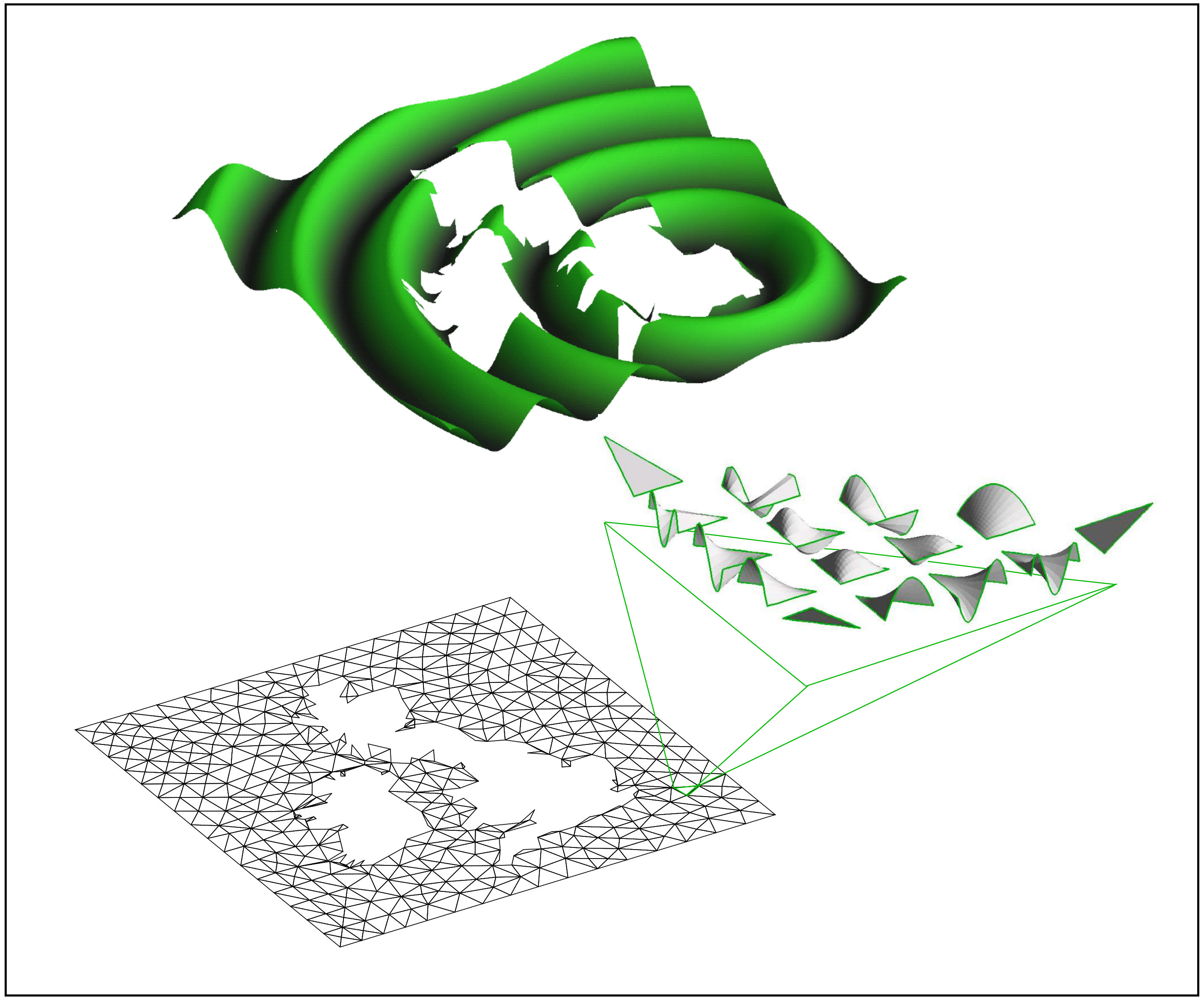
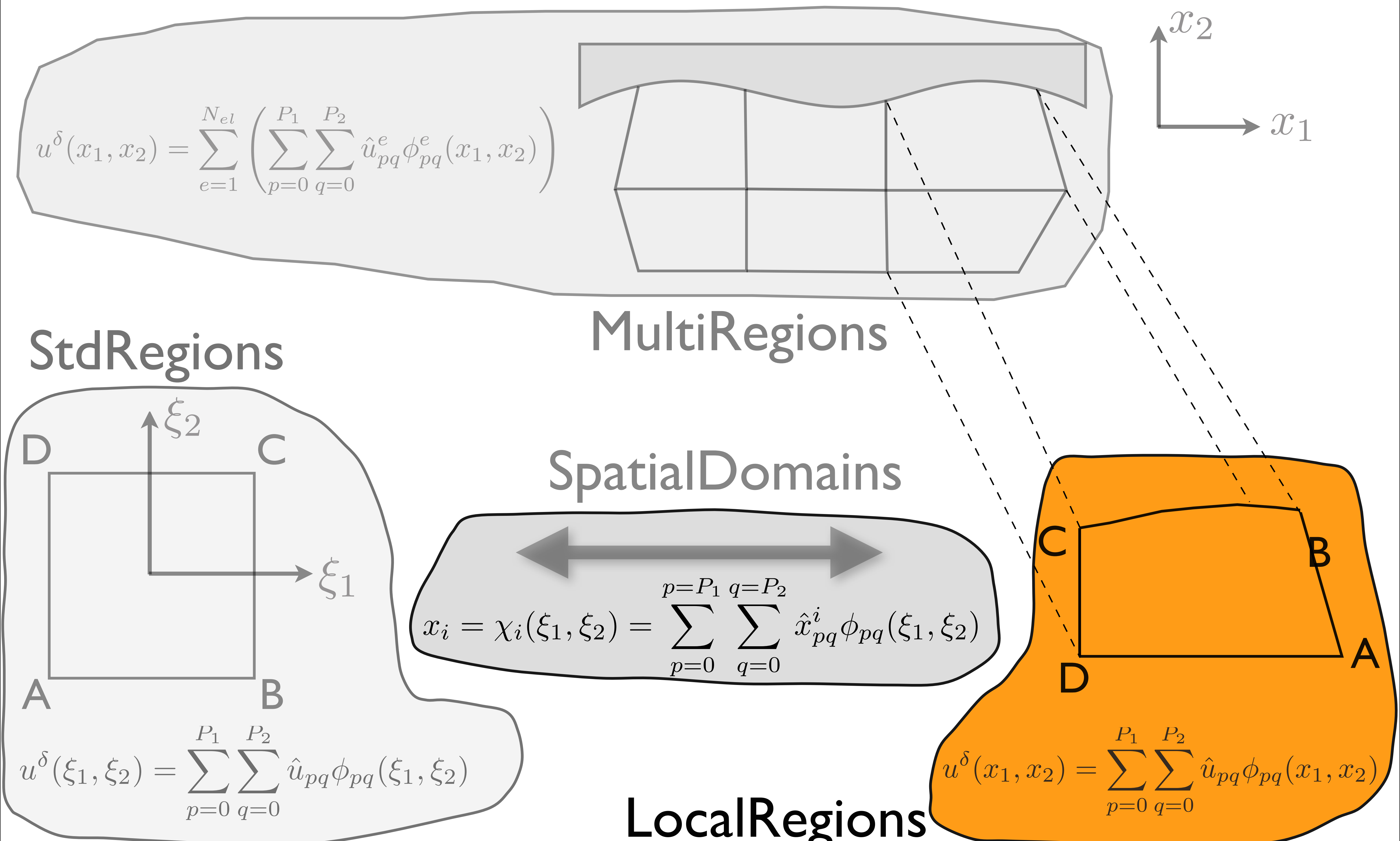
Expansions in Local Regions



The big picture



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Outline

- 2D Integration
    - Metrics (Jacobian)
- 2D Differentiation
    - Metrics
- Elemental projection
- Global Assembly

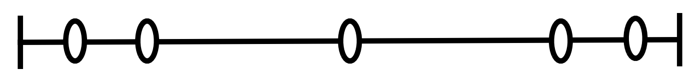
Integration:Gaussian Quadrature

$$\int_{-1}^{1}(1-\xi)^{\alpha}(1+\xi)^{\beta}f(\xi)d\xi = \sum_{p=0}^{Q-1}\boxed{w_p^{\alpha,\beta}}f(\boxed{\xi_p}) + R(f)$$

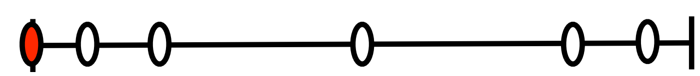
$$w_i^{\alpha,\beta} = \int_{-1}^{1}(1-\xi)^{\alpha}(1+\xi)^{\beta}h_p(\xi)d\xi\,,$$

$$R(f) = \int_{-1}^{1}(1-\xi)^{\alpha}(1+\xi)^{\beta}s(\xi)r(\xi)d\xi\,,$$

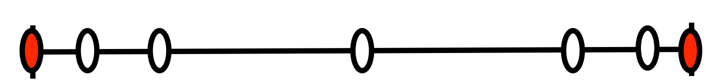
- *Gauss quadrature* is obtained if $k=1$. Gauss quadrature use zeros which lie inside the domain, $-1<\xi_p<1$ for $p=0,\ldots,Q-1$. Gauss quadrature is exact for $f(\xi)$ being a polynomial order $P \le 2Q-1$;



- *Gauss-Radau quadrature* is given by $k=2$. This quadrature uses one point at the end of the domain, usually $\xi_0=-1$. The rest lie inside the domain, $-1<\xi_p<1$ for $p=1,\ldots,Q-1$. Gauss-Radau quadrature is exact for $f(\xi)$ being a polynomial order $P \le 2Q-2$;



- *Gauss-Lobatto quadrature* is defined by $k=3$. Here both end-points of the domain are used, $\xi_0=-1$ and $\xi_{Q-1}=1$. The remaining zeros lie inside the domain, $-1<\xi_i<1$ for $i=1,\ldots,Q-2$. Gauss-Lobatto quadrature is exact for $f(\xi)$ being a polynomial order $P \le 2Q-3$.



Polylib at http://www.nektar.info

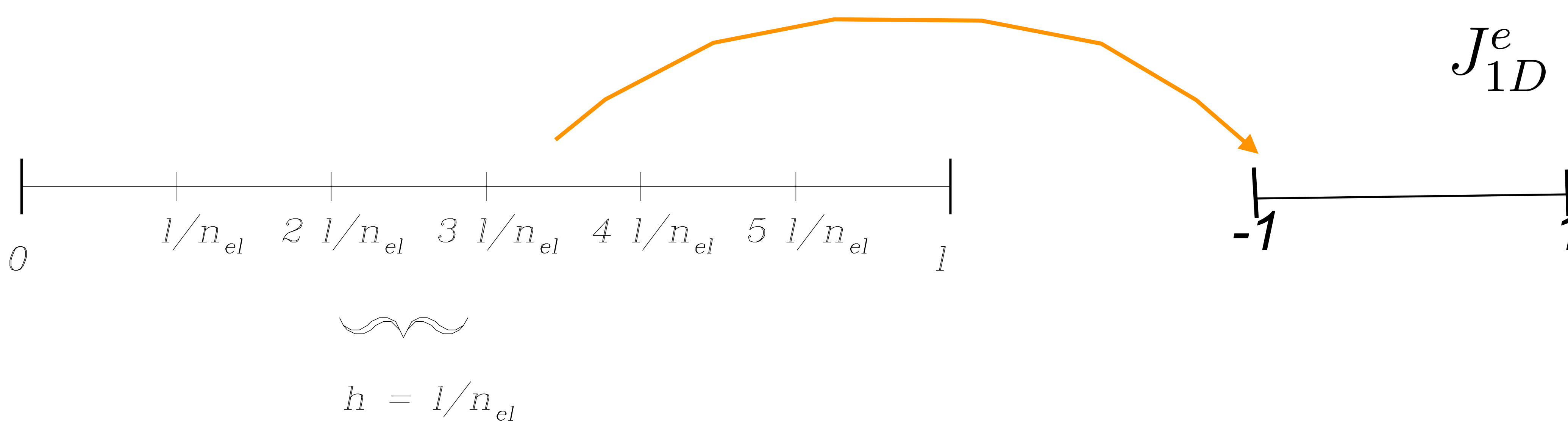
Course Notes: Section 1.4.1.1.

Numerical Integration: general segment

$$\int_{\Omega^e} f(x)\, dx\,,$$

Elemental mapping $\displaystyle \int_{\Omega^e} f(x)\, dx\,, = \int_{-1}^{1} J_{1D}^e\, f(\xi)\, d\xi\,, \qquad J_{1D}^e = \partial x / \partial \xi$

$$J_{1D}^e = h_e/2.$$



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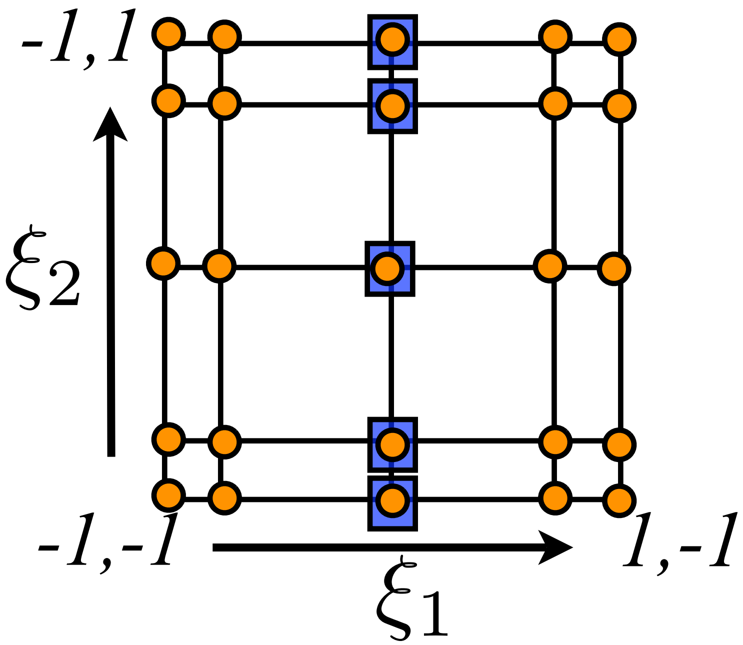
Integration: Standard Quadrilateral

StdRegions::StdQuadExp::Integral()

$$\int_{Q^2} u(\xi_1,\xi_2)\; d\xi_1\; d\xi_2 = \int_{-1}^{1} \left\{ \int_{-1}^{1} u(\xi_1,\xi_2)\Big|_{\xi_2} \; d\xi_1 \right\} d\xi_2 .$$

$$\int_{Q^2} u(\xi_1,\xi_2)\, d\xi_1\, d\xi_2 \simeq \sum_{j=0}^{Q_2-1} w_j \left\{ \sum_{i=0}^{Q_1-1} w_i u(\xi_{1i},\xi_{2j}) \right\}$$

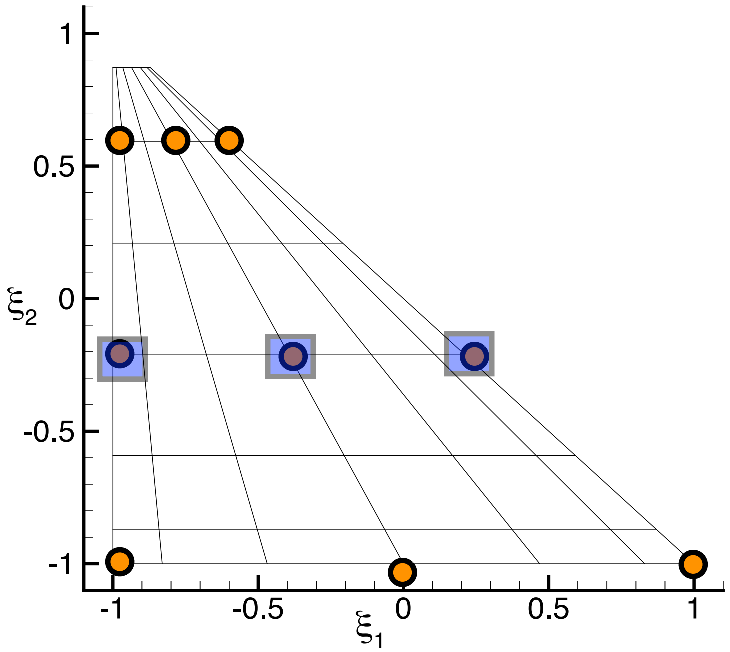


$$Q_1 = Q_2 = 5$$

Course Notes: Section 3.1.1.1

Integration: Standard Triangle

StdRegions::StdTriExp::Integral()



$$\int_{\mathcal{T}^2} u(\xi_1,\xi_2)\, d\xi_1 d\xi_2 = \int_{-1}^{1}\int_{-1}^{-\xi_2} u(\xi_1,\xi_2)\, d\xi_1 d\xi_2$$

$$= \int_{-1}^{1}\int_{-1}^{1} u(\eta_1,\eta_2) \left|\frac{\partial(\xi_1,\xi_2)}{\partial(\eta_1,\eta_2)}\right| \, d\eta_1 d\eta_2,$$

$$\frac{\partial(\xi_1,\xi_2)}{\partial(\eta_1,\eta_2)} = \left(\frac{1-\eta_2}{2}\right).$$

$$\eta_1 = 2\frac{(1+\xi_1)}{(1-\xi_2)} - 1, \qquad \eta_2 = \xi_2.$$

$$\int_{-1}^{1}\int_{-1}^{1} u(\eta_1,\eta_2)\left(\tfrac{1-\eta_2}{2}\right) d\eta_1 d\eta_2 = \boxed{\sum_{i=0}^{Q_1-1} w_i} \left\{ \boxed{\sum_{j=0}^{Q_2-1} w_j \; u(\eta_{1i},\eta_{2j}) \left(\tfrac{1-\eta_{2j}}{2}\right)} \right\}$$

Weighted quadrature: $\int_{-1}^{1} (1-\xi)^{\alpha}(1+\xi)^{\beta} u(\xi) d\xi = \sum_{i=0}^{Q-1} w^{\alpha,\beta} u(\xi_i^{\alpha,\beta}),$

$$\int_{-1}^{1}\int_{-1}^{1} u(\eta_1,\eta_2)\left(\tfrac{1-\eta_2}{2}\right) d\eta_1 d\eta_2 = \sum_{i=0}^{Q_1-1} w_i^{0,0} \left\{ \sum_{j=0}^{Q_2-1} \hat{w}_j^{1,0} u(\eta_{1i},\eta_{2j}) \right\} \qquad \hat{w}_j^{1,0} = \frac{w_j^{1,0}}{2}.$$

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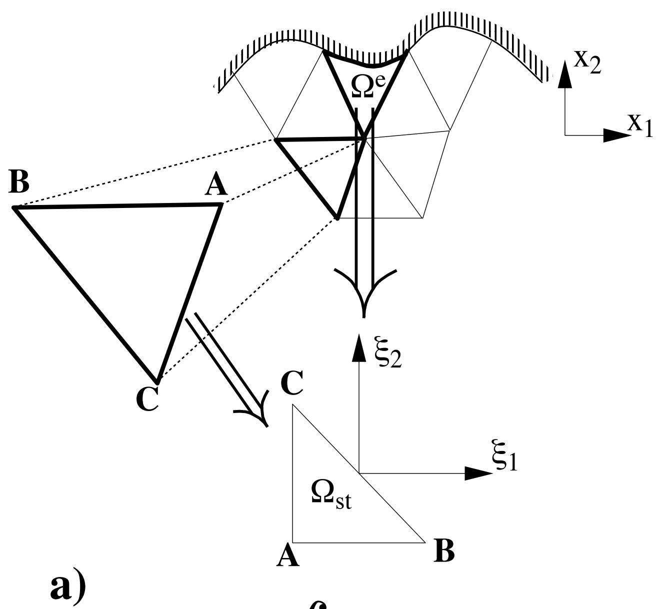
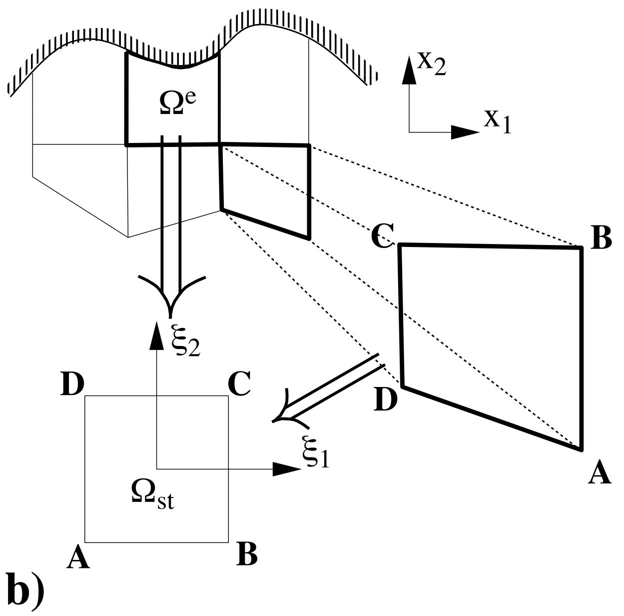
Course Notes: Section 3.1.1.2.

Mapping metrics: Jacobian



$$\chi_1(\xi_1, \xi_2)$$
$$\chi_2(\xi_1, \xi_2)$$



$$\int_{\Omega^e} u(x_1, x_2)\ dx_1\ dx_2 = \int_{\Omega_{st}} u(\xi_1, \xi_2) |J_{2D}|\ d\xi_1\ d\xi_2,$$

Jacobian: $$J_{2D} = \begin{vmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} \end{vmatrix} = \frac{\partial x_1}{\partial \xi_1}\frac{\partial x_2}{\partial \xi_2} - \frac{\partial x_1}{\partial \xi_2}\frac{\partial x_2}{\partial \xi_1}.$$

Note Jacobian requires differentation with respect to standard coordintates

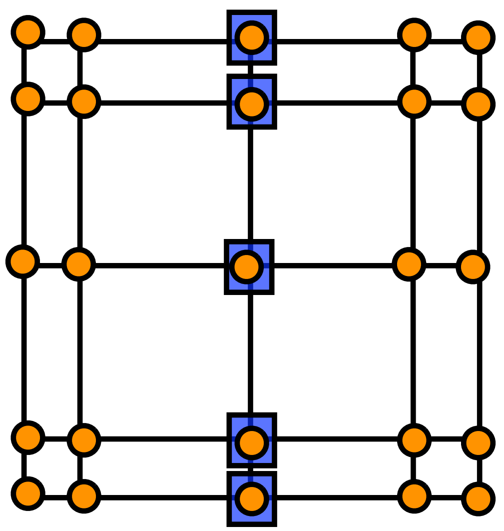
Course Notes: Section 3.1.3.1 & 3.1.3.3

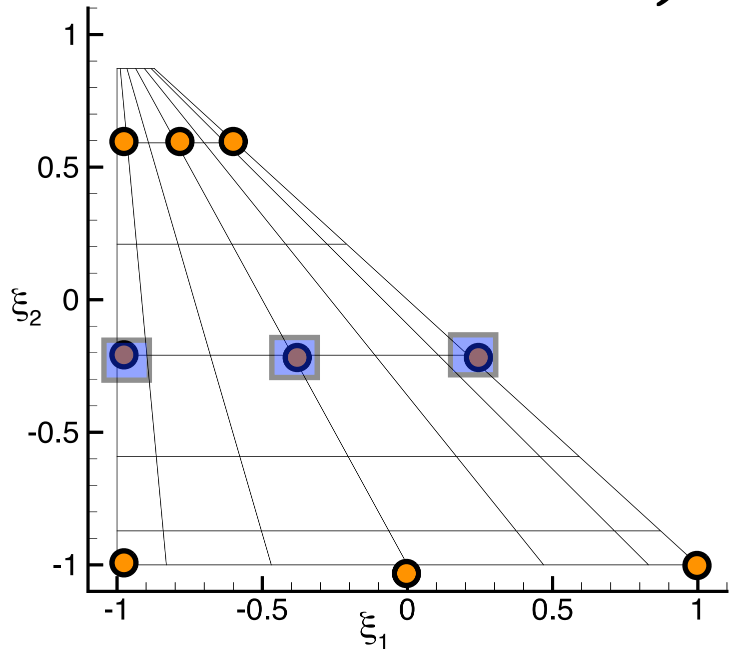
Integration: General 2D Elements

LocalRegions::QuadExp::Integral()
LocalRegions::TriExp::Integral()

$$\int_{\Omega^e} u(x_1, x_2)\ dx_1\ dx_2 = \int_{\Omega_{st}} u(\xi_1, \xi_2)|J_{2D}|\ d\xi_1\ d\xi_2,$$

$$\int_{\Omega^e} u(x_1, x_2)\, dx_1\, dx_2 \simeq \sum_{j=0}^{Q_2-1} w_j \left\{ \sum_{i=0}^{Q_1-1} w_i\, u(\xi_{1i}, \xi_{2j})|J_{2D}|(\xi_{1i}, \xi_{2j}) \right\}$$





Course Notes: Section 3.1.3.3

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Outline

- 2D Integration
  - Metrics (Jacobian)
- 2D Differentiation
  - Metrics
- Elemental projection
- Global Assembly

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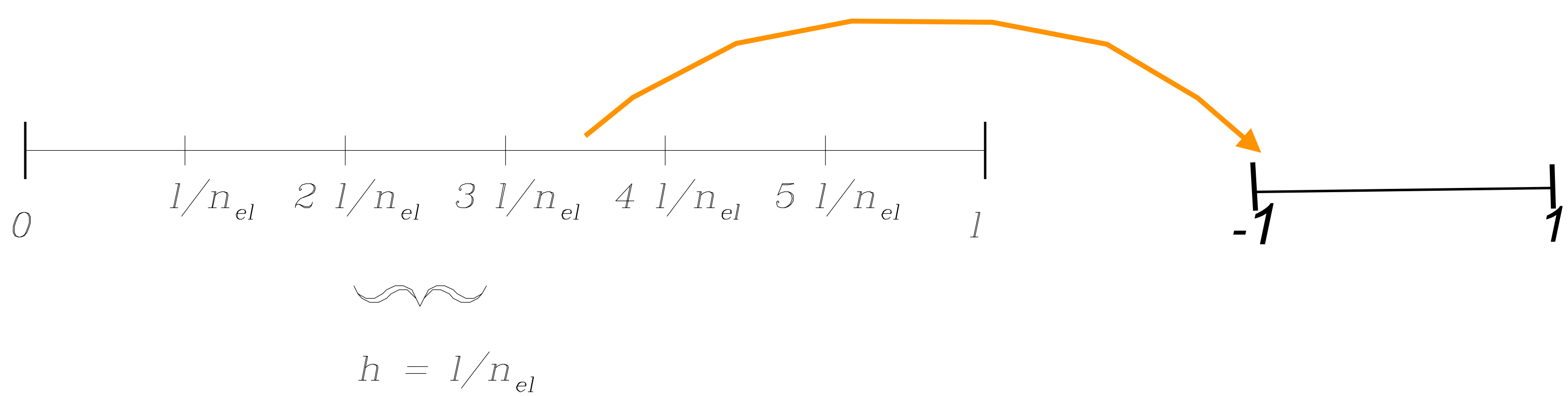
Numerical Differentiation

Elemental representation:

$$u^\delta(x) = \sum_{p=0}^{P} \hat{u}_p \phi_p(\chi^{-1}) = \sum_{p=0}^{P} \hat{u}_p \phi_p(\xi), \quad = \sum_{p=0}^{P} u(\xi_p) h_p(\xi)$$

Chain rule:

$$\frac{du^\delta}{dx} = \boxed{\frac{du^\delta}{d\xi}} \frac{d\xi}{dx}$$



$$\frac{d\xi}{dx} = \frac{h}{2}$$

Collocation differentiation:

$$\boxed{\left.\frac{du(\xi)}{d\xi}\right|_{\xi=\xi_i}} = \sum_{j=0}^{Q-1} d_{ij} u(\xi_j)\,, \quad d_{ij} = \left.\frac{dh_j(\xi)}{d\xi}\right|_{\xi=\xi_i}$$

Collocation property:

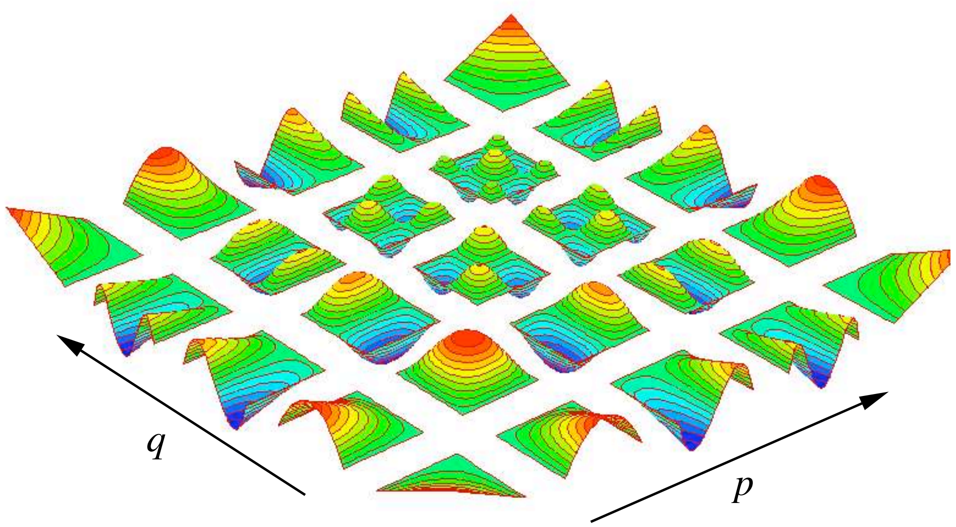
$$\mathrm{u}(\xi)\frac{du}{dx} = \sum_{p=0}^{P} u(x_p) h_p(\xi) \cdot \sum_{p=0}^{P} \frac{du}{d\xi}(\xi_p) h_p(\xi) \simeq \sum_{p=0}^{P} u(\xi_p) \frac{du}{d\xi}(x_p) h_p(\xi)$$

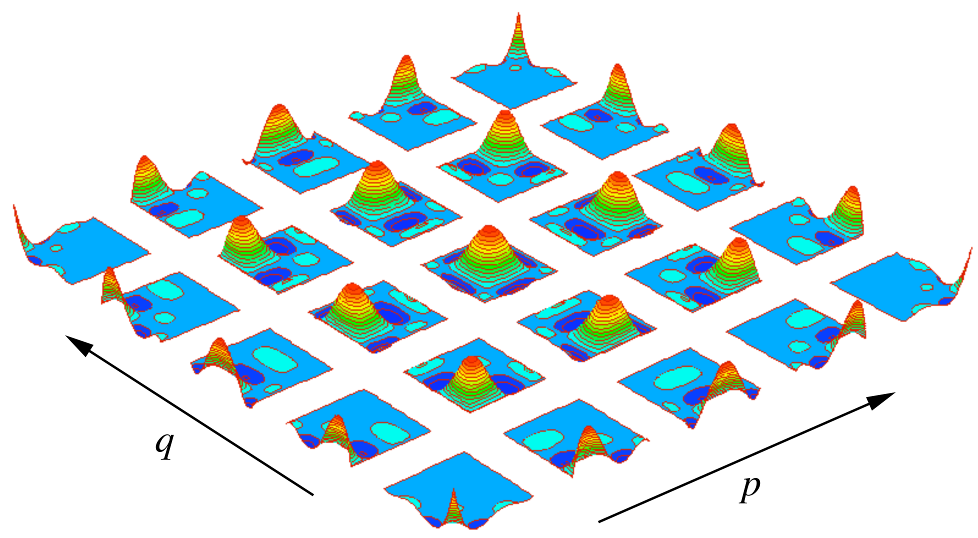
Course Notes: Section 1.4.2, Appendix C

Numerical Differentiation: Standard 2D regions

$$u^\delta(\xi_1,\xi_2) = \sum_{p=0}^{P_1}\sum_{q=0}^{P_2} \hat{u}_{pq}\phi_{pq}(\xi_1,\xi_2), \quad \longleftrightarrow \quad u^\delta(\xi_1,\xi_2) = \sum_{p=0}^{Q_1-1}\sum_{q=0}^{Q_2-1} u_{pq}\; h_p(\xi_1)h_q(\xi_2),$$

$$u_{pq} = u^\delta(\xi_{1p},\xi_{2q}),$$





Course Notes: Section 3.1.2.1

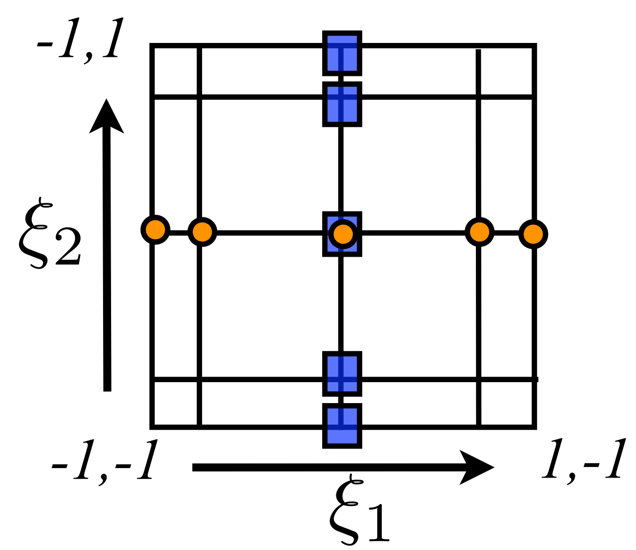
Numerical Differentiation: Standard Quad

StdRegions::StdQuadExp::PhysDeriv()

$$u^\delta(\xi_1,\xi_2) = \sum_{p=0}^{P_1}\sum_{q=0}^{P_2} \hat{u}_{pq}\phi_{pq}(\xi_1,\xi_2), \quad \longleftrightarrow \quad u^\delta(\xi_1,\xi_2) = \sum_{p=0}^{Q_1-1}\sum_{q=0}^{Q_2-1} u_{pq}\, h_p(\xi_1) h_q(\xi_2),$$

$$u_{pq} = u^\delta(\xi_{1p},\xi_{2q}),$$

$$\frac{\partial u^\delta}{\partial \xi_1}(\xi_1,\xi_2) = \sum_{p=0}^{P_1}\sum_{q=0}^{P_2} u_{pq}\frac{dh_p(\xi_1)}{d\xi_1} h_q(\xi_2).$$



$$\frac{\partial u^\delta}{\partial \xi_1}(\xi_{1i},\xi_{2j}) = \sum_{p=0}^{P_1}\sum_{q=0}^{P_2}\left\{ u_{pq}\left.\frac{dh_p(\xi_1)}{d\xi_1}\right|_{\xi_{1i}} \delta_{qj}\right\} = \sum_{p=0}^{P_1} u_{pj}\left.\frac{dh_p(\xi_1)}{d\xi_1}\right|_{\xi_{1i}}$$

$$\boxed{\frac{\partial u^\delta}{\partial \xi_2}(\xi_{1i},\xi_{2j})} = \sum_{q=0}^{P_2} u_{iq}\left.\frac{dh_q(\xi_2)}{d\xi_2}\right|_{\xi_{2j}}.$$

Course Notes: Section 3.1.2.1

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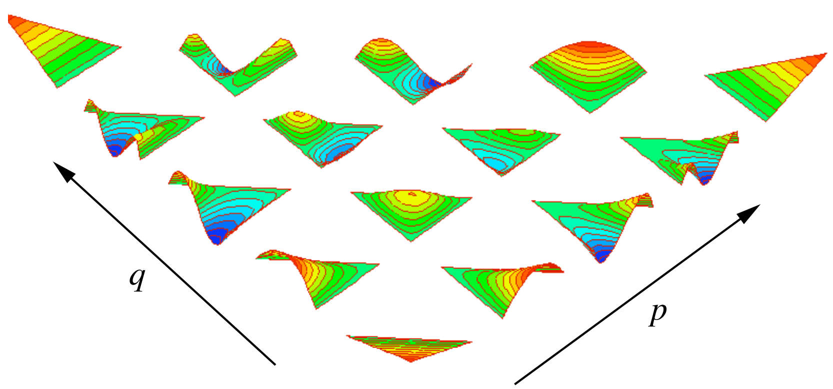
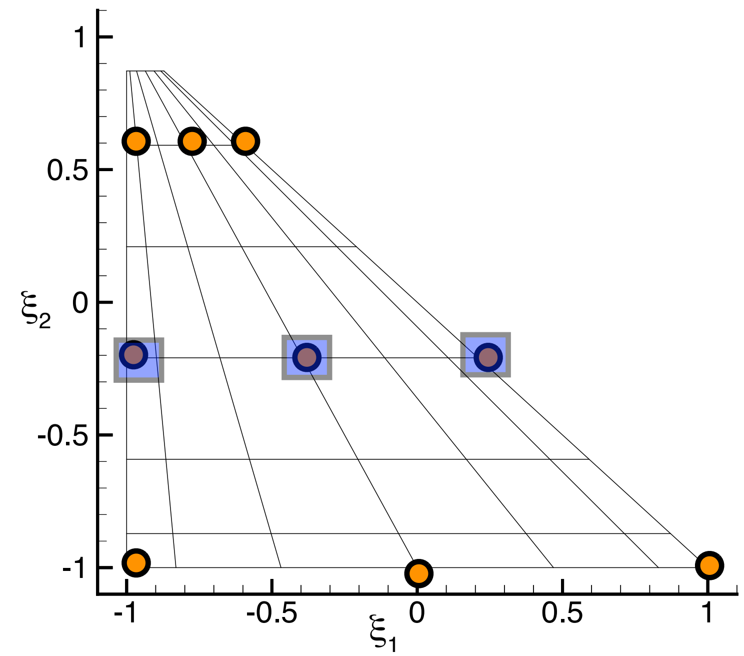
Numerical Differentiation: Standard Tri

StdRegions::StdTriExp::PhysDeriv()

$$u^\delta(\xi_1,\xi_2) = \sum_{p,q}^{P_1,P_2} \hat{u}_{pq}\ \phi_{pq}(\eta_1,\eta_2) \quad \Longleftrightarrow \quad \sum_{p=0}^{P_1}\sum_{q=0}^{P_2} u_{pq} h_p(\eta_1) h_q(\eta_2), \qquad u_{pq} = u^\delta(\eta_{1p},\eta_{2q}),$$





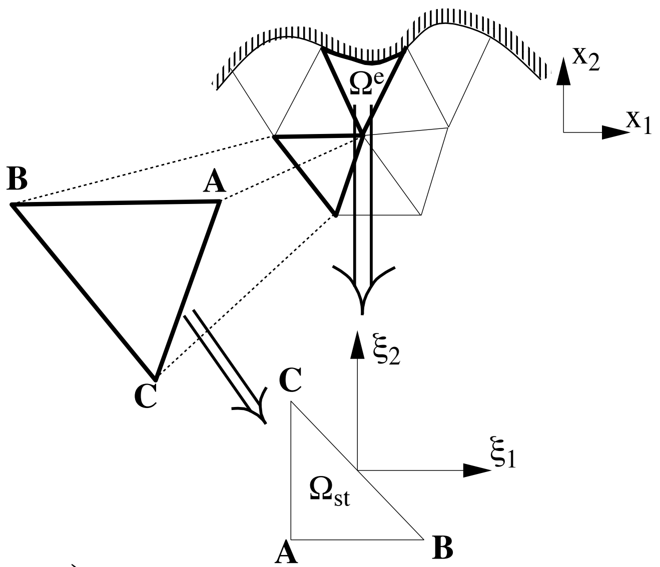
$$\eta_1 = \frac{2(1+\xi_1)}{(1-\xi_2)} - 1, \qquad \eta_2 = \xi_2,$$

Apply chain rule:
$$\begin{pmatrix} \dfrac{\partial}{\partial \xi_1} \\[2ex] \dfrac{\partial}{\partial \xi_2} \end{pmatrix} = \begin{pmatrix} \dfrac{2}{(1-\eta_2)} \dfrac{\partial}{\partial \eta_1} \\[2ex] 2\dfrac{(1+\eta_1)}{(1-\eta_2)} \dfrac{\partial}{\partial \eta_1} \quad + \quad \dfrac{\partial}{\partial \eta_2} \end{pmatrix}$$

Course Notes: Section 3.1.2.1

Numerical Differentiation: Local Regions

LocalRegions::QuadExp/TriExp::PhysDeriv()



a)

$$\nabla = \begin{bmatrix} \dfrac{\partial}{\partial x_1} \\[2ex] \dfrac{\partial}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \xi_1}{\partial x_1}\dfrac{\partial}{\partial \xi_1} + \dfrac{\partial \xi_2}{\partial x_1}\dfrac{\partial}{\partial \xi_2} \\[2ex] \dfrac{\partial \xi_1}{\partial x_2}\dfrac{\partial}{\partial \xi_1} + \dfrac{\partial \xi_2}{\partial x_2}\dfrac{\partial}{\partial \xi_2} \end{bmatrix}$$

$\chi_1(\xi_1, \xi_2)$
$\chi_2(\xi_1, \xi_2)$

$$dx_1(\xi_1, \xi_2) = \frac{\partial x_1}{\partial \xi_1} d\xi_1 + \frac{\partial x_1}{\partial \xi_2} d\xi_2$$

$$\begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x_1}{\partial \xi_1} & \dfrac{\partial x_1}{\partial \xi_2} \\[2ex] \dfrac{\partial x_2}{\partial \xi_1} & \dfrac{\partial x_2}{\partial \xi_2} \end{bmatrix} \begin{bmatrix} d\xi_1 \\ d\xi_2 \end{bmatrix}$$

$$\begin{bmatrix} d\xi_1 \\ d\xi_2 \end{bmatrix} = \frac{1}{J} \begin{bmatrix} \boxed{\dfrac{\partial x_2}{\partial \xi_2}} & \boxed{-\dfrac{\partial x_1}{\partial \xi_2}} \\[2ex] \boxed{-\dfrac{\partial x_2}{\partial \xi_1}} & \boxed{\dfrac{\partial x_1}{\partial \xi_1}} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix},$$

$$d\xi_1(x_1, x_2) = \frac{\partial \xi_1}{\partial x_1} dx_1 + \frac{\partial \xi_1}{\partial x_2} dx_2$$

$$\begin{bmatrix} d\xi_1 \\ d\xi_2 \end{bmatrix} = \begin{bmatrix} \boxed{\dfrac{\partial \xi_1}{\partial x_1}} & \boxed{-\dfrac{\partial \xi_1}{\partial x_2}} \\[2ex] \boxed{\dfrac{\partial \xi_2}{\partial x_1}} & \boxed{\dfrac{\partial \xi_2}{\partial x_2}} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}.$$

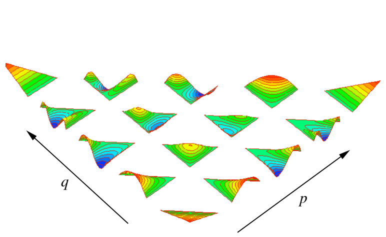
$$\frac{\partial \xi_1}{\partial x_1} = \frac{1}{J}\frac{\partial x_2}{\partial \xi_2}, \qquad \frac{\partial \xi_1}{\partial x_2} = -\frac{1}{J}\frac{\partial x_1}{\partial \xi_2}, \qquad \frac{\partial \xi_2}{\partial x_1} = -\frac{1}{J}\frac{\partial x_2}{\partial \xi_1}, \qquad \frac{\partial \xi_2}{\partial x_2} = \frac{1}{J}\frac{\partial x_1}{\partial \xi_1}.$$

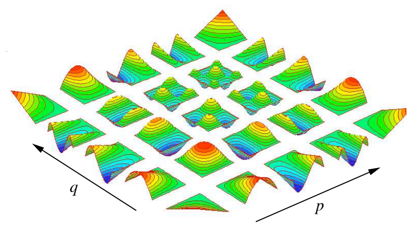
Course Notes: Section 3.1.3.4

Elemental forward transform

LocalRegions::QuadExp/TriExp::FwdTrans()

Expansion: $u^\delta = \sum_{p,q} \hat{u}_{pq}\phi_{pq}$





Approximation:

$$u^\delta(\xi_1,\xi_2) - u(\xi_1,\xi_2) = R(u)$$

$$\left(\sum_{pq} \hat{u}_{pq}\phi_{pq}(\xi_1,\xi_2)\right) - u(\xi_1,\xi_2) = R(u).$$

Method of weighted residual:

$$(v, \sum_{p,q} \hat{u}_{pq}\phi_{pq}) - (v,u) = \cancel{(v, R(u))},$$

$$(a,b) = \int_{\Omega^e} ab\, d\mathbf{x}$$

$$(v, \sum_{p,q} \hat{u}_{pq}\phi_{pq}) = (v,u).$$

Course Notes: Sections 1.1 & 3.1.5.3

Elemental forward transform

LocalRegions::QuadExp/TriExp::FwdTrans()

$$\boxed{(v, \sum_{p,q} \hat{u}_{pq}\phi_{pq}) = (v, u).}$$

Galerkin weight: $v(\xi_1, \xi_2) = \phi_{rs}(\xi_1, \xi_2).$ $\Rightarrow$ $(\phi_{rs}, \sum_{p,q} \hat{u}_{pq}\phi_{pq}) = (\phi_{rs}, u)$

$$\boxed{\sum_{p,q} (\phi_{rs}, \phi_{pq})\hat{u}_{pq} = (\phi_{rs}, u),}$$

$$\mathbf{M}[i][j] = \int_{\Omega^e} \phi_{i(r,s)}(x_1, x_2)\phi_{j(p,q)}(x_1, x_2)dx_1dx_2 \qquad \hat{\mathbf{u}}[i] = \hat{u}_{i(p,q)}$$

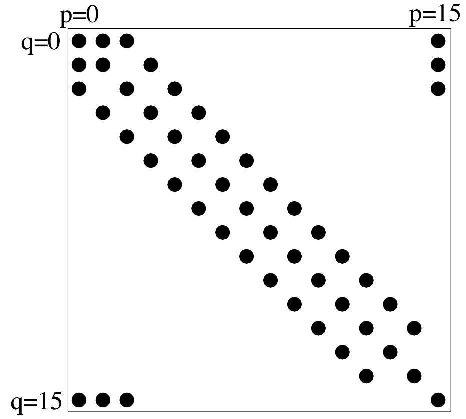
$$\mathbf{f}[i] = \int_{\Omega^e} \phi_{i(p,q)} u(x_1, x_2)dx_1dx_2$$

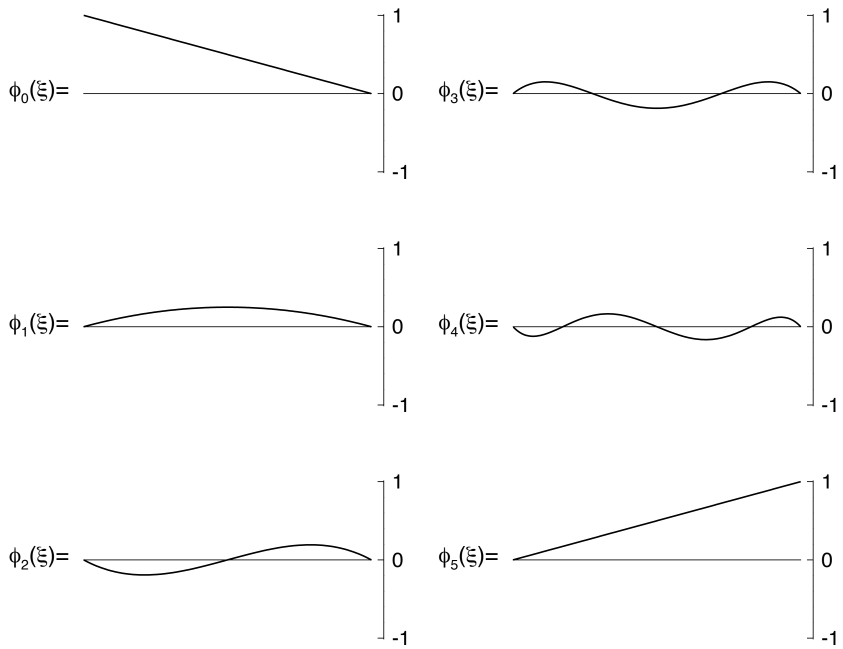
$$\boxed{\hat{\mathbf{u}} = \mathbf{M}^{-1}\mathbf{f}}$$

Course Notes: Section 3.1.5.3

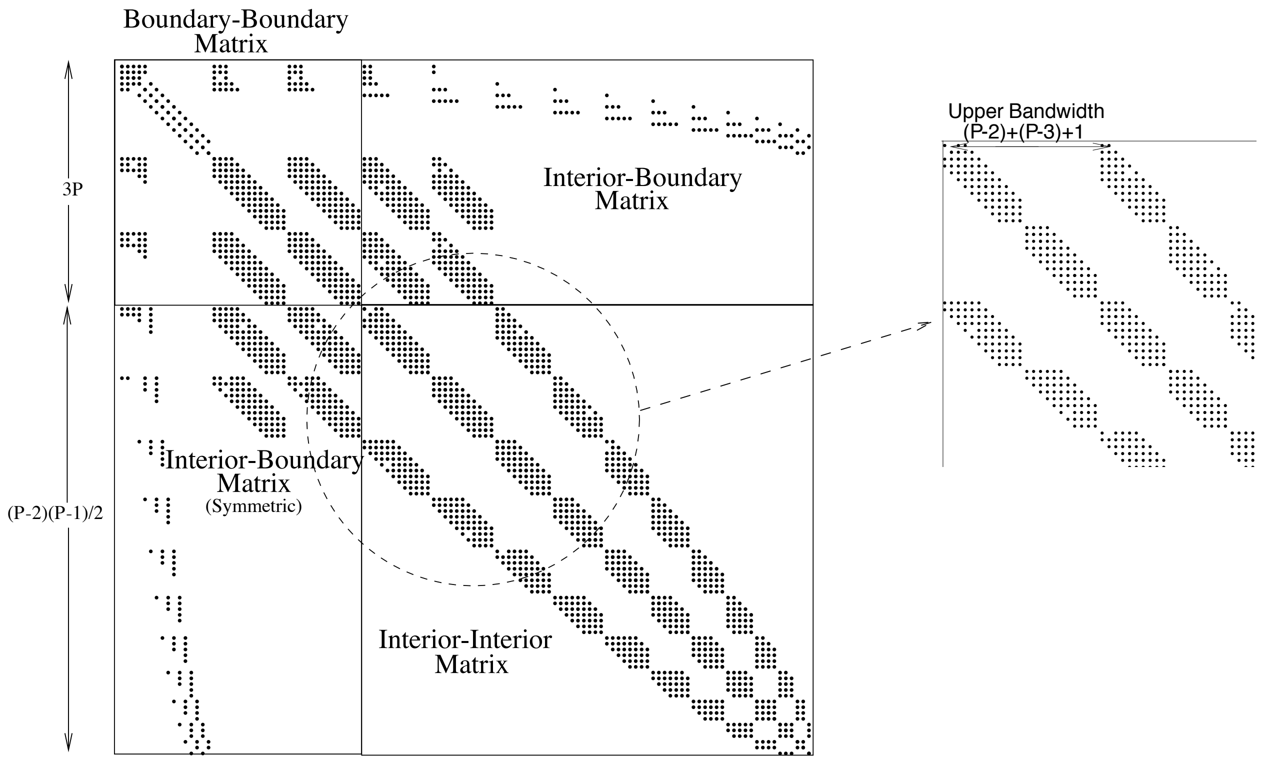
Elemental Mass matrix structure

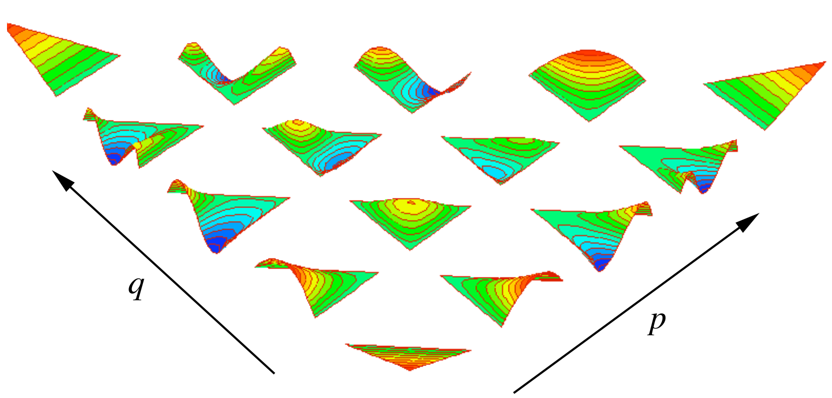
Segment expansion P =15





Triangular expansion P=14





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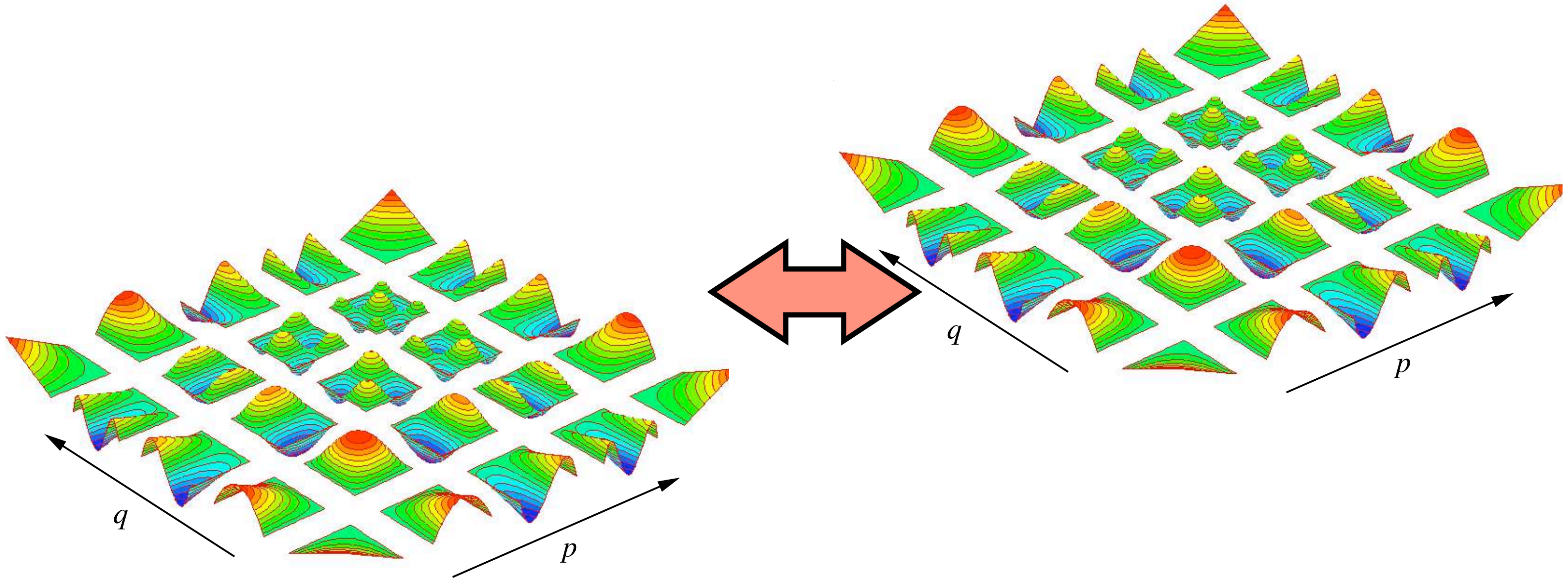
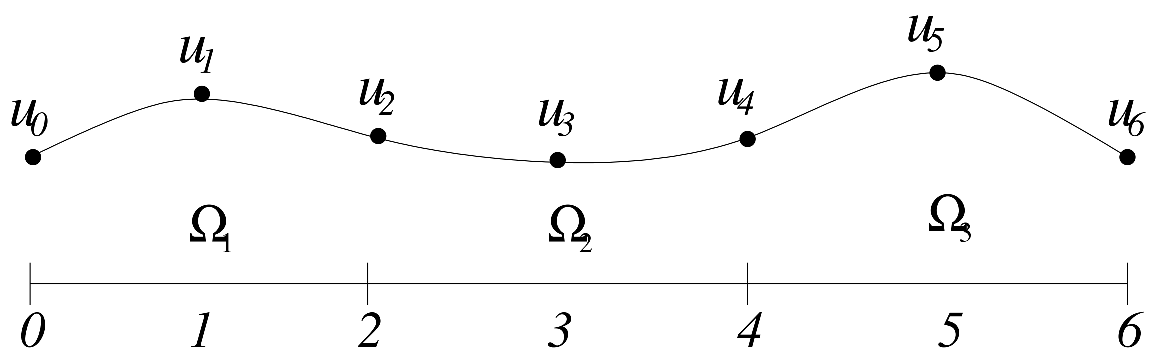
Global Assembly

- Classical continuous Galerkin FEM
- 2nd order PDE: $\nabla^2 u - \lambda u = -f$
  - C_0 continuity sufficient.
  - Use boundary interior decomposition to make continuous expansion.



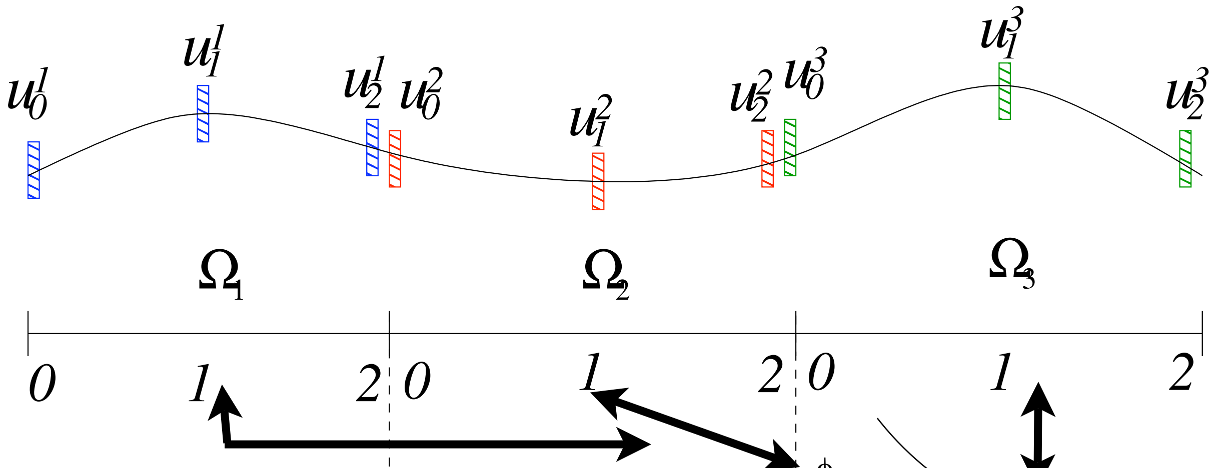
Global Assembly



$u(\mathbf{x}) = \sum_{i=0}^{6} u_i \Phi(\mathbf{x})$ Global Numbering

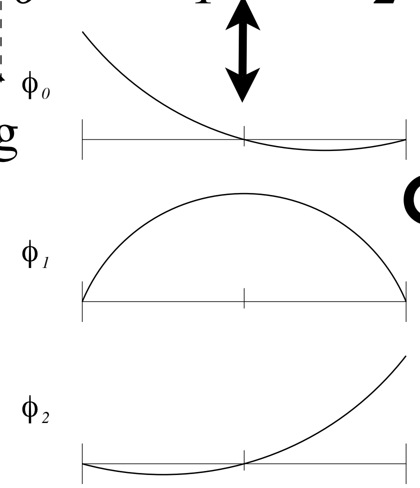
$$\tilde{\mathbf{u}}_l = \mathcal{A}\tilde{\mathbf{u}}_g$$

$$\tilde{\mathbf{u}}_l = \begin{bmatrix} u_0^1 \\ u_1^1 \\ u_2^1 \\ u_0^2 \\ u_1^2 \\ u_2^2 \\ u_0^3 \\ u_1^3 \\ u_2^3 \end{bmatrix} = \mathcal{A}\tilde{\mathbf{u}}_g = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u}_0 \\ \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \\ \tilde{u}_4 \\ \tilde{u}_5 \\ \tilde{u}_6 \end{bmatrix}$$

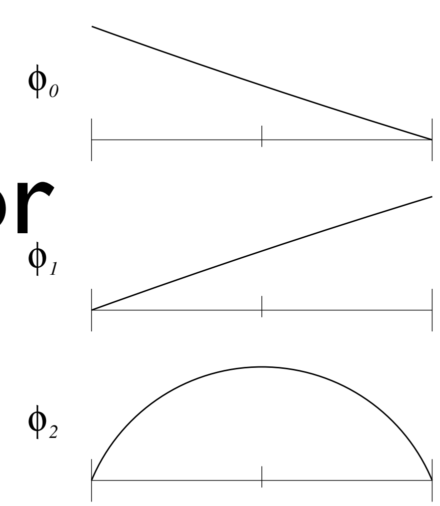


Local Numbering

$u(\mathbf{x}) = \sum_{e=0}^{2}\sum_{i=2}^{6} u_i^e \phi(\xi)$



or



$$\tilde{\mathbf{i}}_g = \mathcal{A}^T\tilde{\mathbf{i}}_l$$

$$\mathcal{A}^T\mathcal{A} \neq \mathbf{I}$$

Course Notes: Sections 1.3.1.4 & 3.2.1

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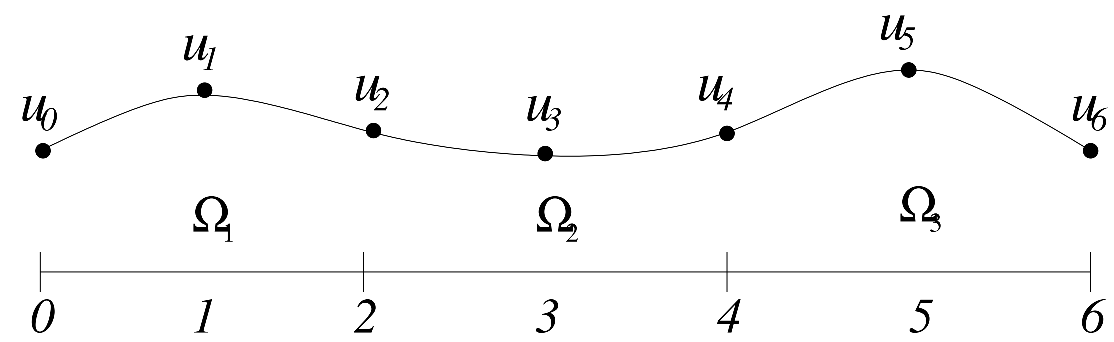
Global Assembly

$$
\tilde{\mathbf{u}}_l = \begin{bmatrix} u_0^1 \\ u_1^1 \\ u_2^1 \\ u_0^2 \\ u_1^2 \\ u_2^2 \\ u_0^3 \\ u_1^3 \\ u_2^3 \end{bmatrix} = \mathcal{A}\tilde{\mathbf{u}}_g = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u}_0 \\ \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \\ \tilde{u}_4 \\ \tilde{u}_5 \\ \tilde{u}_6 \end{bmatrix}
$$



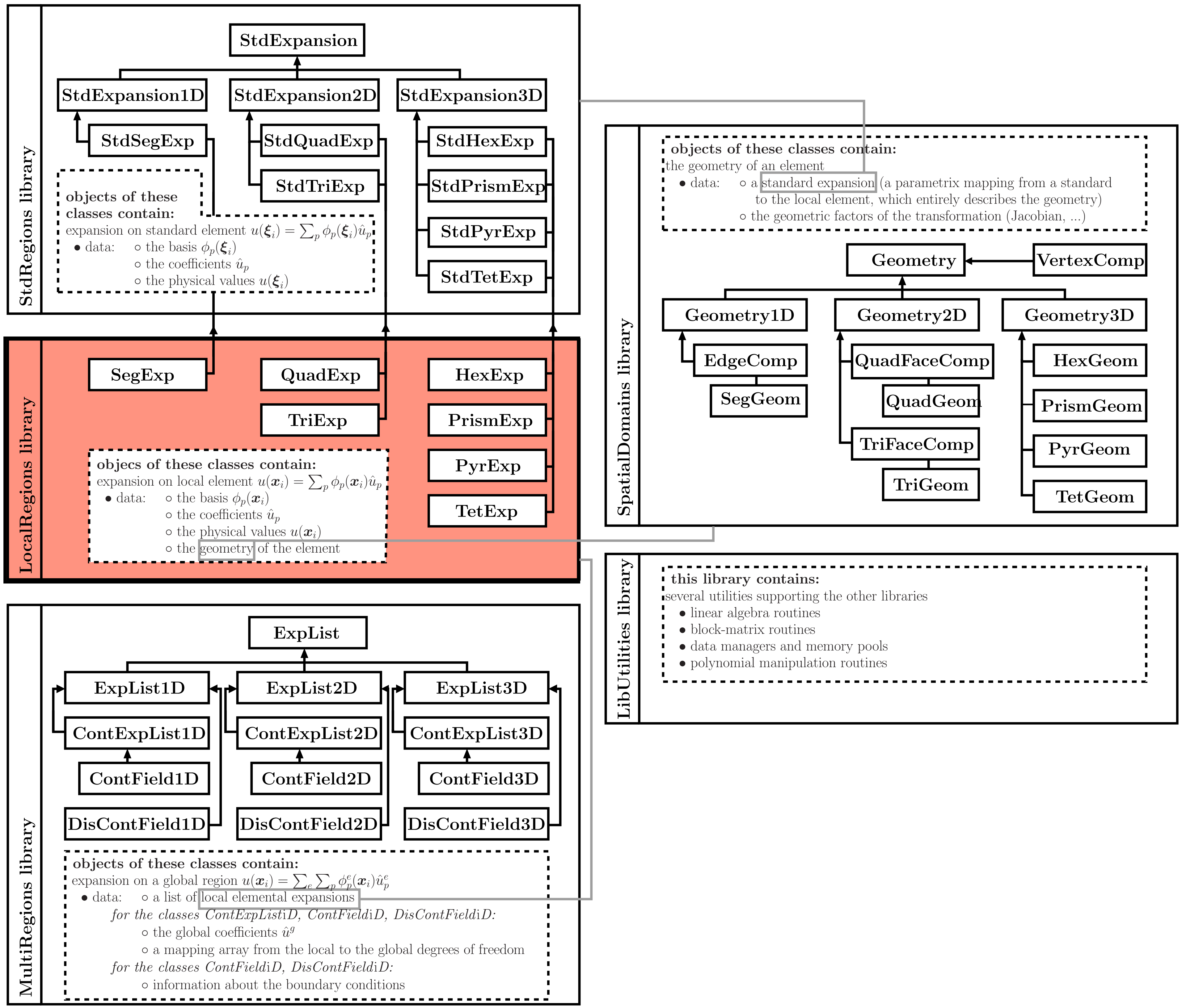
Global Numbering

$$
\mathrm{map}[1][p] = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathrm{map}[2][p] = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \quad \mathrm{map}[3][p] = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}
$$

$$
\begin{aligned}
&\text{for } e = 1 : N_{el} \\
&\quad \text{for } p = 0 : P^e \\
&\quad\quad \tilde{\mathbf{u}}^e[p] = \tilde{\mathbf{u}}_g[\mathrm{map}[e][p]] \\
&\quad \text{end} \\
&\text{end}
\end{aligned}
$$

$$
\begin{aligned}
&\text{for } e = 1 : N_{el} \\
&\quad \text{for } p = 0 : P^e \\
&\quad\quad \text{for } q = 0 : P^e \\
&\quad\quad\quad \mathbf{M}_g[\mathrm{map}[e][p]][\mathrm{map}[e][q]] = \\
&\quad\quad\quad\quad \mathbf{M}_g[\mathrm{map}[e][p]][\mathrm{map}[e][q]] + \mathbf{M}^e[p][q] \\
&\quad\quad \text{end} \\
&\quad \text{end} \\
&\text{end}
\end{aligned}
$$

Nektar++ code



Nektar++