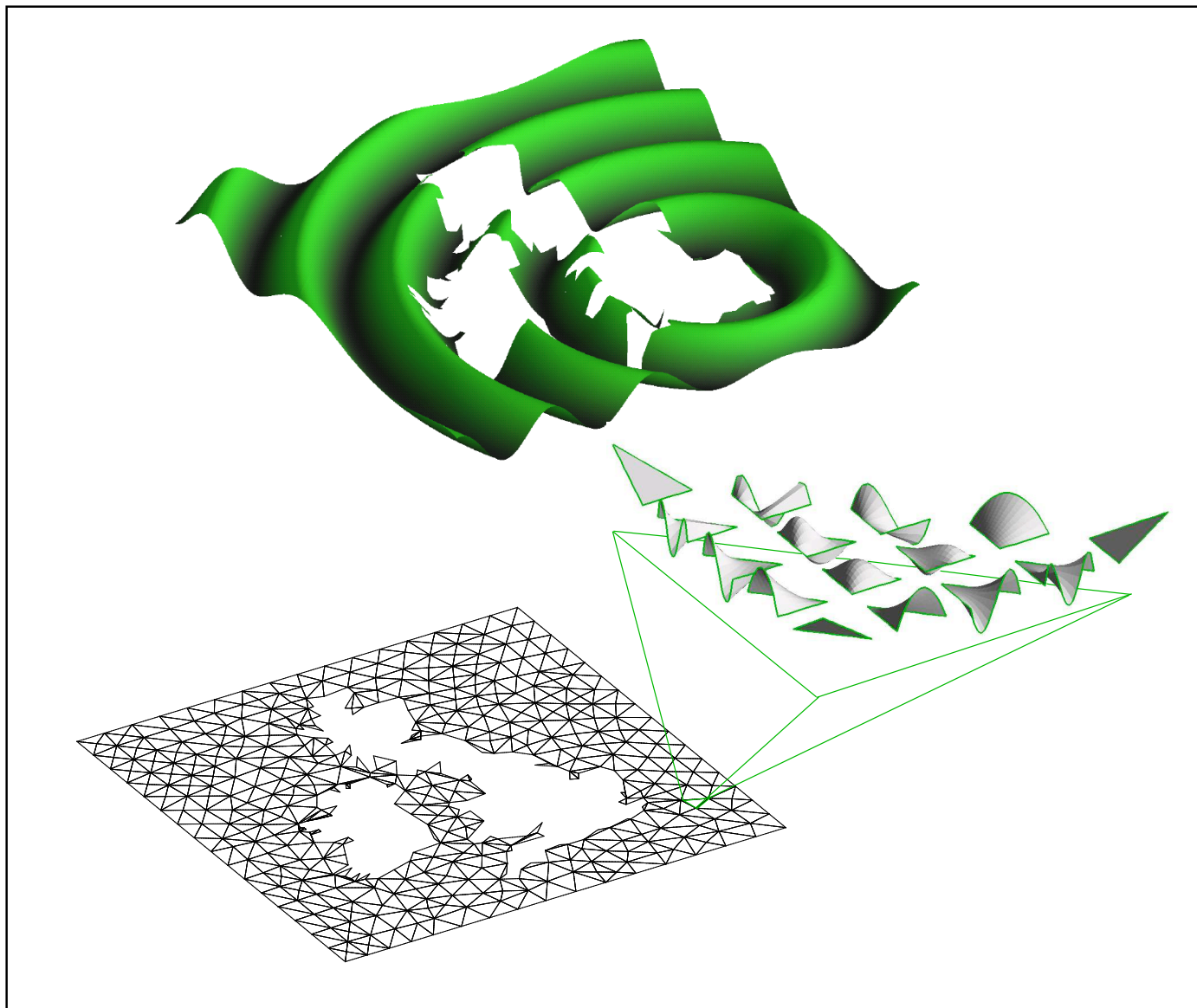
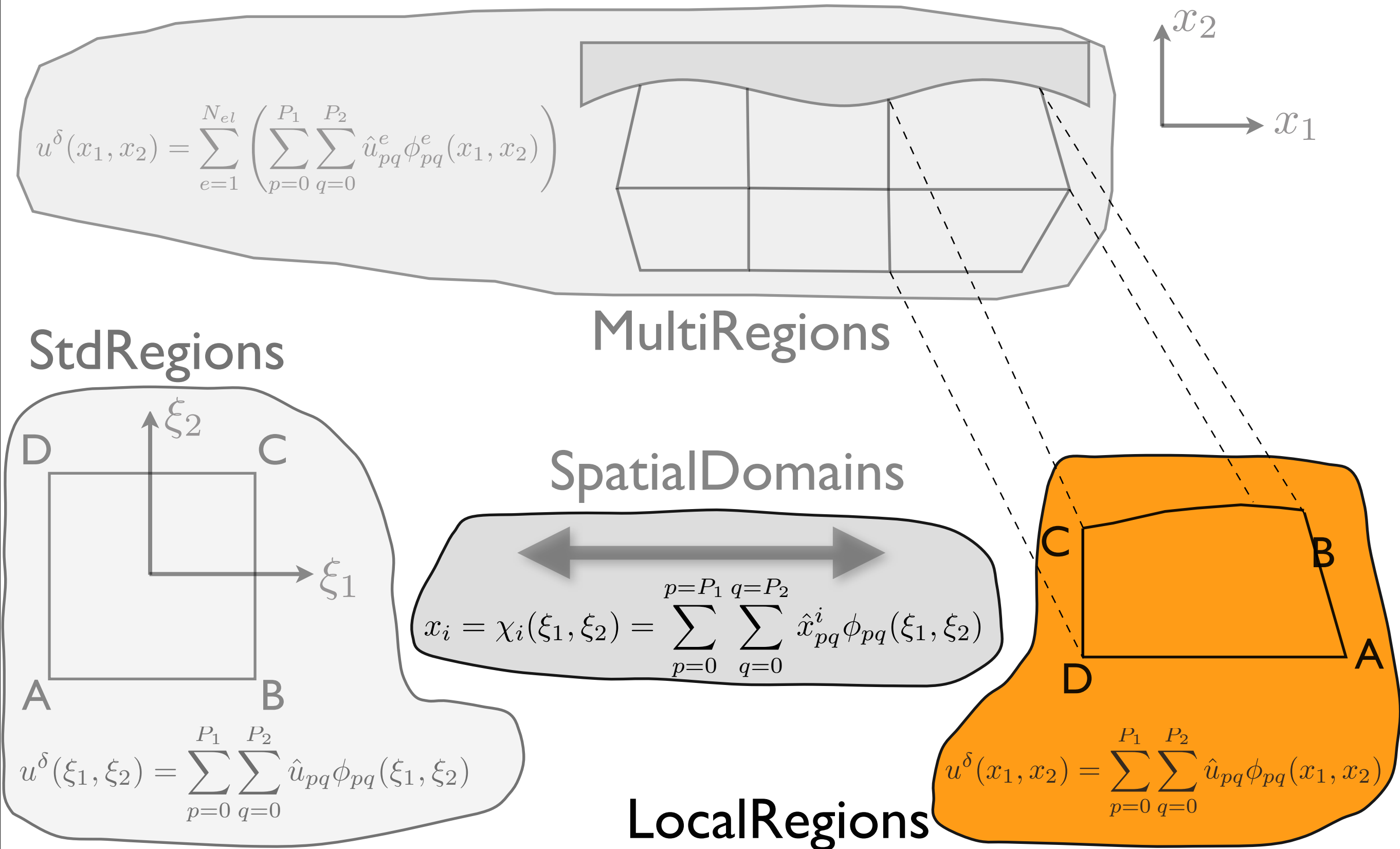


Expansions in Local Regions



The big picture



Outline

- 2D Integration
 - Metrics (Jacobian)
- 2D Differentiation
 - Metrics
- Elemental projection
- Global Assembly

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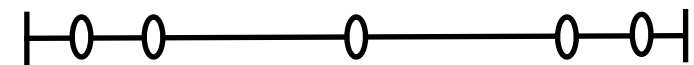
Integration: Gaussian Quadrature

$$\int_{-1}^1 (1 - \xi)^\alpha (1 + \xi)^\beta f(\xi) d\xi = \sum_{p=0}^{Q-1} w_p^{\alpha, \beta} f(\xi_p) + R(f)$$

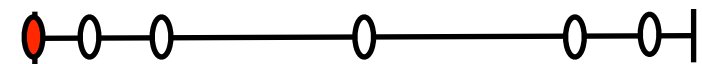
$$w_i^{\alpha, \beta} = \int_{-1}^1 (1 - \xi)^\alpha (1 + \xi)^\beta h_p(\xi) d\xi,$$

$$R(f) = \int_{-1}^1 (1 - \xi)^\alpha (1 + \xi)^\beta s(\xi) r(\xi) d\xi,$$

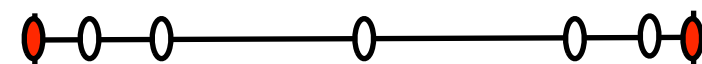
- *Gauss quadrature* is obtained if $k = 1$. Gauss quadrature uses zeros which lie inside the domain, $-1 < \xi_p < 1$ for $p = 0, \dots, Q - 1$. Gauss quadrature is exact for $f(\xi)$ being a polynomial order $P \leq 2Q - 1$;



- *Gauss-Radau quadrature* is given by $k = 2$. This quadrature uses one point at the end of the domain, usually $\xi_0 = -1$. The rest lie inside the domain, $-1 < \xi_p < 1$ for $p = 1, \dots, Q - 1$. Gauss-Radau quadrature is exact for $f(\xi)$ being a polynomial order $P \leq 2Q - 2$;



- *Gauss-Lobatto quadrature* is defined by $k = 3$. Here both end-points of the domain are used, $\xi_0 = -1$ and $\xi_{Q-1} = 1$. The remaining zeros lie inside the domain, $-1 < \xi_i < 1$ for $i = 1, \dots, Q - 2$. Gauss-Lobatto quadrature is exact for $f(\xi)$ being a polynomial order $P \leq 2Q - 3$.



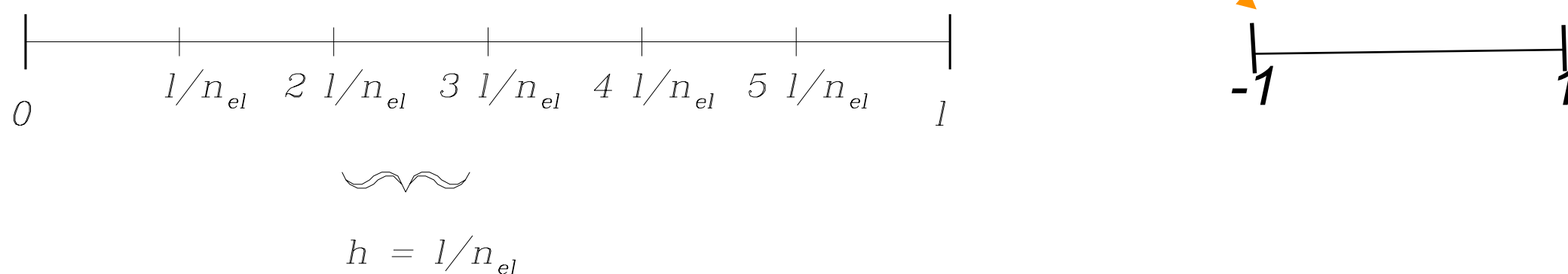
Polylib at <http://www.nektar.info>

Numerical Integration: general segment

$$\int_{\Omega^e} f(x) dx ,$$

Elemental mapping $\int_{\Omega^e} f(x) dx , = \int_{-1}^1 J_{1D}^e f(\xi) d\xi , \quad J_{1D}^e = \partial x / \partial \xi$

$$J_{1D}^e = h_e/2.$$

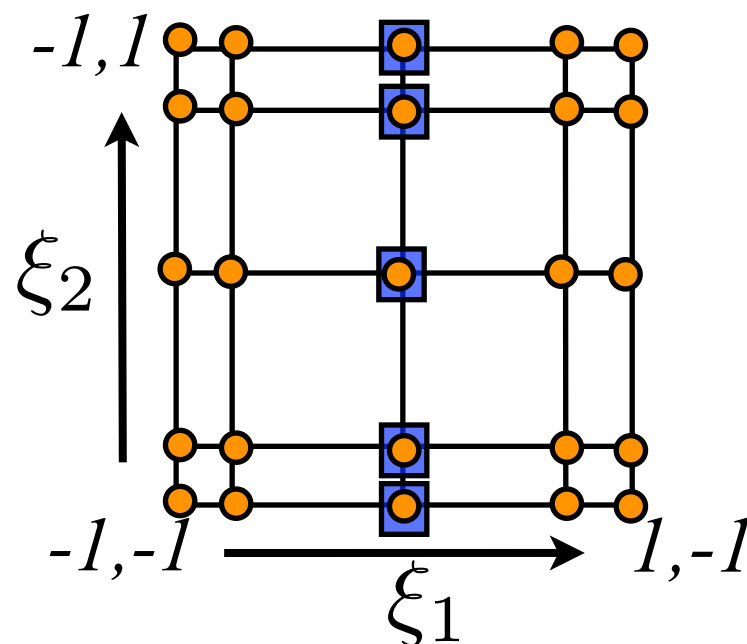


Integration: Standard Quadrilateral

StdRegions::StdQuadExp::Integral()

$$\int_{Q^2} u(\xi_1, \xi_2) d\xi_1 d\xi_2 = \int_{-1}^1 \left\{ \int_{-1}^1 u(\xi_1, \xi_2) \Big|_{\xi_2} d\xi_1 \right\} d\xi_2.$$

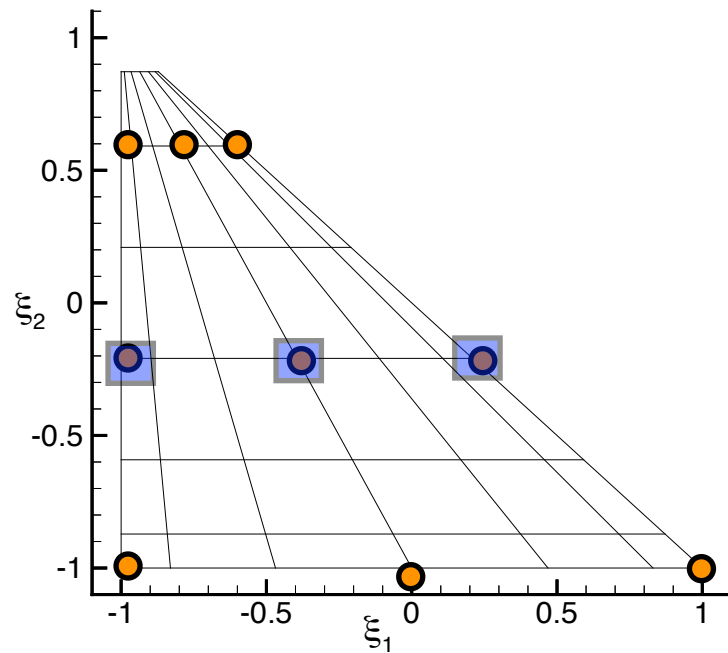
$$\int_{Q^2} u(\xi_1, \xi_2) d\xi_1 d\xi_2 \simeq \sum_{j=0}^{Q_2-1} w_j \left\{ \sum_{i=0}^{Q_1-1} w_i u(\xi_{1i}, \xi_{2j}) \right\}$$



$$Q_1 = Q_2 = 5$$

Integration: Standard Triangle

StdRegions::StdTriExp::Integral()



$$\begin{aligned}\int_{T^2} u(\xi_1, \xi_2) d\xi_1 d\xi_2 &= \int_{-1}^1 \int_{-1}^{-\xi_2} u(\xi_1, \xi_2) d\xi_1 d\xi_2 \\ &= \int_{-1}^1 \int_{-1}^1 u(\eta_1, \eta_2) \left| \frac{\partial(\xi_1, \xi_2)}{\partial(\eta_1, \eta_2)} \right| d\eta_1 d\eta_2,\end{aligned}$$

$$\frac{\partial(\xi_1, \xi_2)}{\partial(\eta_1, \eta_2)} = \left(\frac{1 - \eta_2}{2} \right).$$

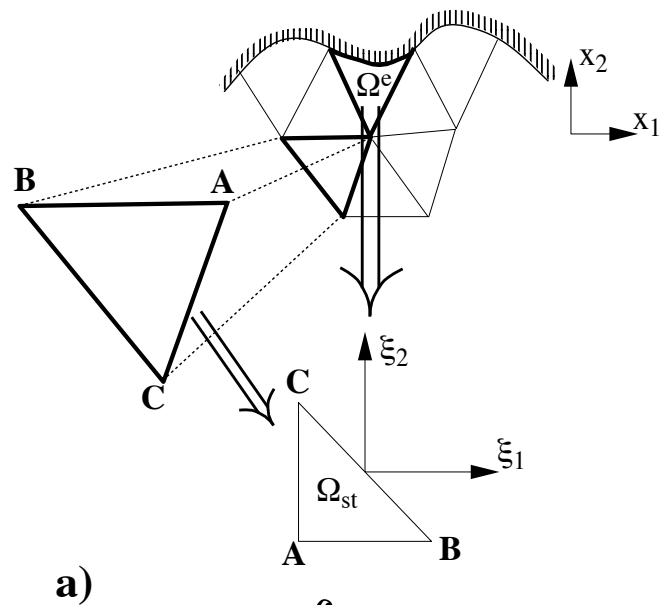
$$\eta_1 = 2 \frac{(1 + \xi_1)}{(1 - \xi_2)} - 1, \quad \eta_2 = \xi_2.$$

$$\int_{-1}^1 \int_{-1}^1 u(\eta_1, \eta_2) \left(\frac{1 - \eta_2}{2} \right) d\eta_1 d\eta_2 = \left[\sum_{i=0}^{Q_1-1} w_i \right] \left\{ \sum_{j=0}^{Q_2-1} w_j u(\eta_{1i}, \eta_{2j}) \left(\frac{1 - \eta_{2j}}{2} \right) \right\}$$

Weighted quadrature: $\int_{-1}^1 (1 - \xi)^\alpha (1 + \xi)^\beta u(\xi) d\xi = \sum_{i=0}^{Q-1} w^{\alpha, \beta} u(\xi_i^{\alpha, \beta}),$

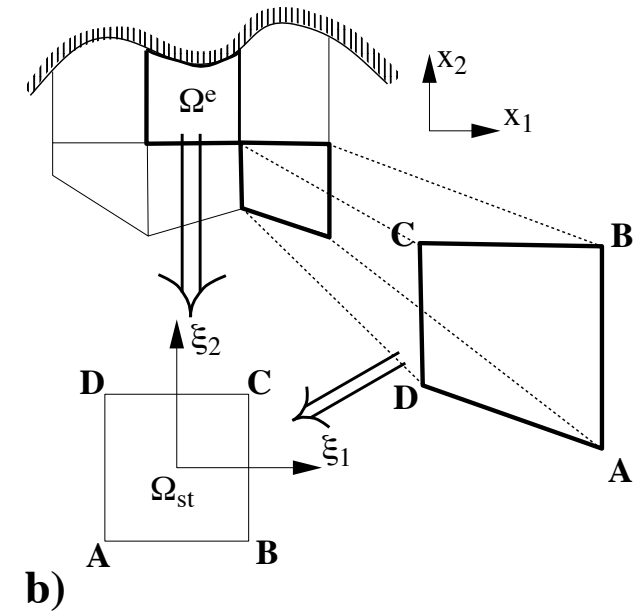
$$\int_{-1}^1 \int_{-1}^1 u(\eta_1, \eta_2) \left(\frac{1 - \eta_2}{2} \right) d\eta_1 d\eta_2 = \sum_{i=0}^{Q_1-1} w_i^{0,0} \left\{ \sum_{j=0}^{Q_2-1} \hat{w}_j^{1,0} u(\eta_{1i}, \eta_{2j}) \right\} \quad \hat{w}_j^{1,0} = \frac{w_j^{1,0}}{2}.$$

Mapping metrics: Jacobian



$$\chi_1(\xi_1, \xi_2)$$

$$\chi_2(\xi_1, \xi_2)$$



$$\int_{\Omega^e} u(x_1, x_2) \, dx_1 \, dx_2 = \int_{\Omega_{st}} u(\xi_1, \xi_2) |J_{2D}| \, d\xi_1 \, d\xi_2,$$

Jacobian:

$$J_{2D} = \begin{vmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} \end{vmatrix} = \frac{\partial x_1}{\partial \xi_1} \frac{\partial x_2}{\partial \xi_2} - \frac{\partial x_1}{\partial \xi_2} \frac{\partial x_2}{\partial \xi_1}.$$

Note Jacobian requires differentiation with respect to standard coordinates

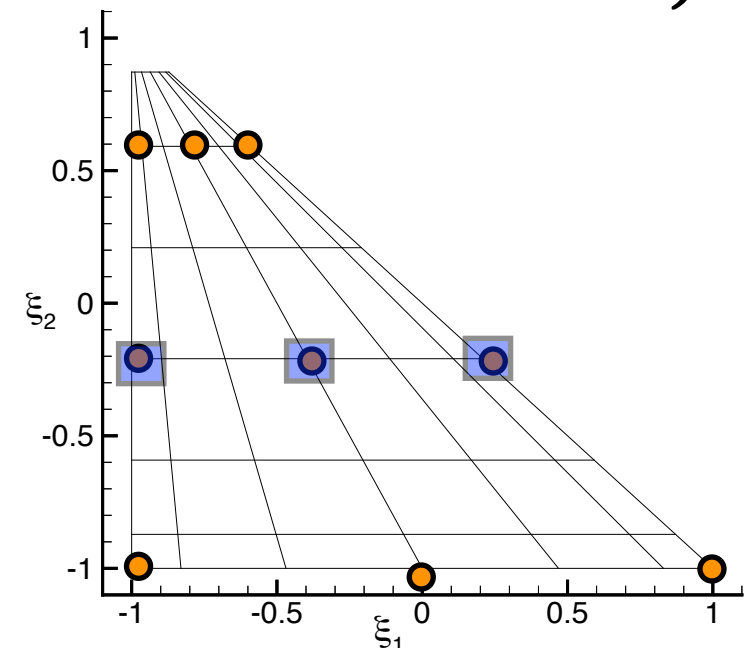
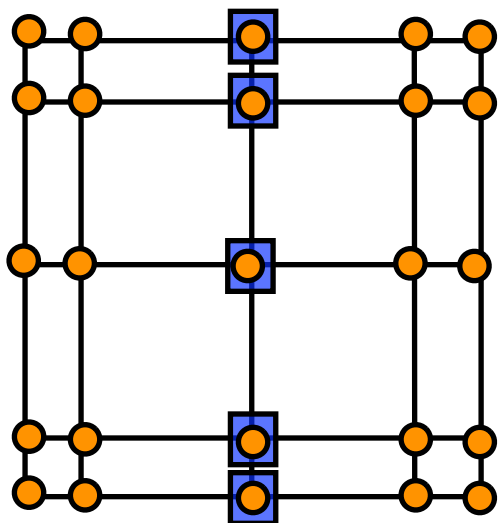
Integration: General 2D Elements

LocalRegions::QuadExp::Integral()

LocalRegions::TriExp::Integral()

$$\int_{\Omega^e} u(x_1, x_2) dx_1 dx_2 = \int_{\Omega_{st}} u(\xi_1, \xi_2) |J_{2D}| d\xi_1 d\xi_2,$$

$$\int_{\Omega^e} u(x_1, x_2) dx_1 dx_2 \simeq \sum_{j=0}^{Q_2-1} w_j \left\{ \sum_{i=0}^{Q_1-1} w_i u(\xi_{1i}, \xi_{2j}) |J_{2D}|(\xi_{1i}, \xi_{2j}) \right\}$$



Course Notes: Section 3.1.3.3

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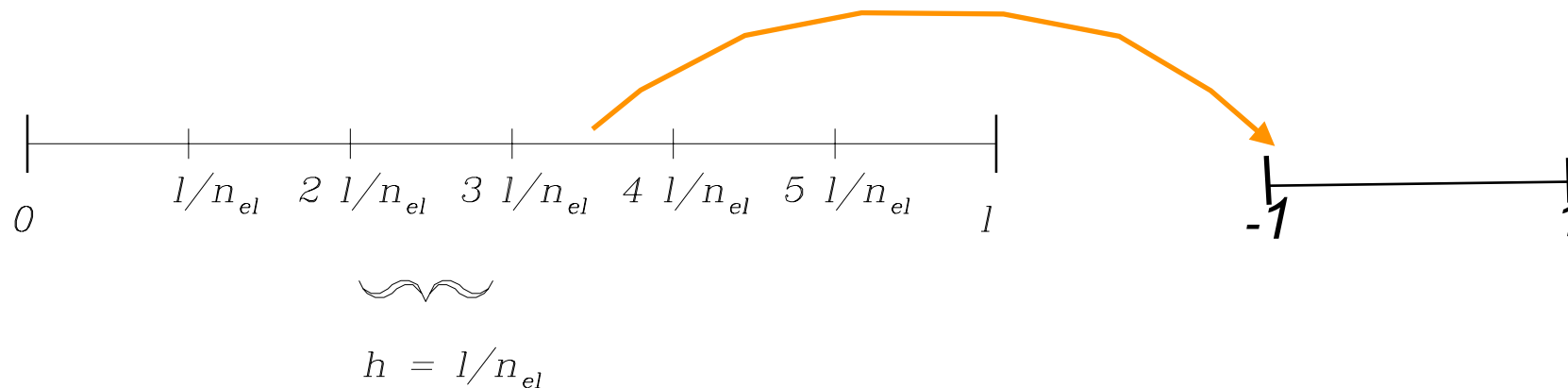
Numerical Differentiation

Elemental
representation:

$$u^\delta(x) = \sum_{p=0}^P \hat{u}_p \phi_p(\chi^{-1}) = \sum_{p=0}^P \hat{u}_p \phi_p(\xi), = \sum_{p=0}^P u(\xi_p) h_p(\xi)$$

Chain rule:

$$\frac{du^\delta}{dx} = \boxed{\frac{du^\delta}{d\xi}} \frac{d\xi}{dx}$$



$$\frac{d\xi}{dx} = \frac{h}{2}$$

Collocation
differentiation:

$$\boxed{\left. \frac{du(\xi)}{d\xi} \right|_{\xi=\xi_i}} = \sum_{j=0}^{Q-1} d_{ij} u(\xi_j), \quad d_{ij} = \left. \frac{dh_j(\xi)}{d\xi} \right|_{\xi=\xi_i}$$

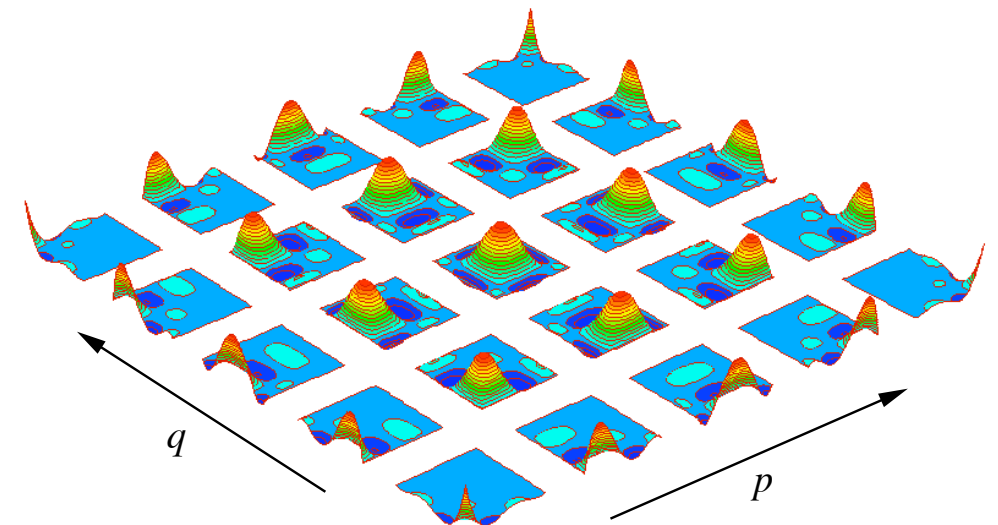
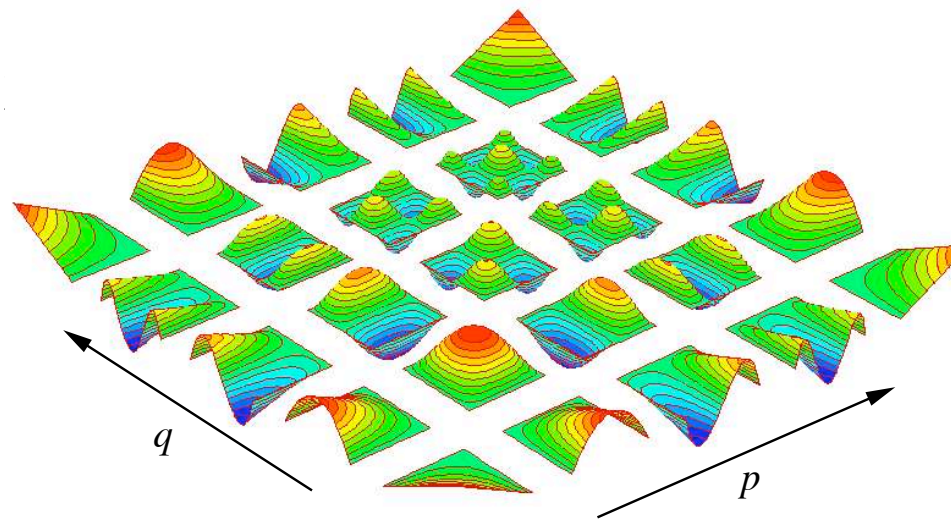
Collocation
property:

$$u(\xi) \frac{du}{dx} = \sum_{p=0}^P u(x_p) h_p(\xi) \cdot \sum_{p=0}^P \frac{du}{d\xi}(\xi_p) h_p(\xi) \simeq \sum_{p=0}^P u(\xi_p) \frac{du}{d\xi}(x_p) h_p(\xi)$$

Numerical Differentiation: Standard 2D regions

$$u^\delta(\xi_1, \xi_2) = \sum_{p=0}^{P_1} \sum_{q=0}^{P_2} \hat{u}_{pq} \phi_{pq}(\xi_1, \xi_2), \quad \longleftrightarrow \quad u^\delta(\xi_1, \xi_2) = \sum_{p=0}^{Q_1-1} \sum_{q=0}^{Q_2-1} u_{pq} h_p(\xi_1) h_q(\xi_2),$$

$$u_{pq} = u^\delta(\xi_{1p}, \xi_{2q}),$$



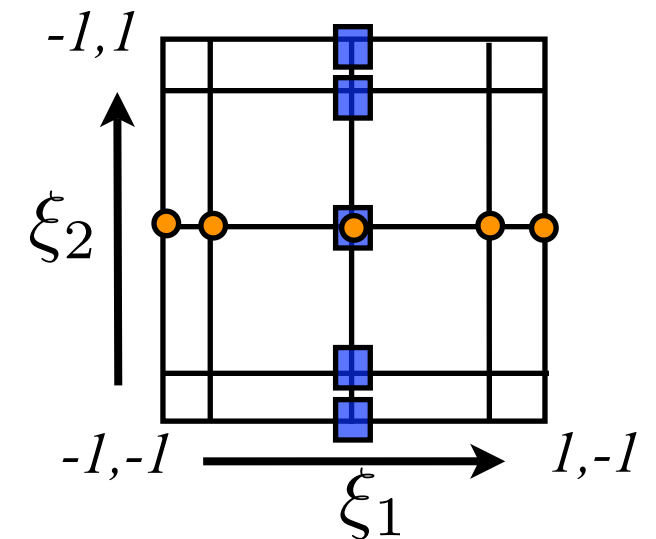
Numerical Differentiation: Standard Quad

StdRegions::StdQuadExp::PhysDeriv()

$$u^\delta(\xi_1, \xi_2) = \sum_{p=0}^{P_1} \sum_{q=0}^{P_2} \hat{u}_{pq} \phi_{pq}(\xi_1, \xi_2), \quad \longleftrightarrow \quad u^\delta(\xi_1, \xi_2) = \sum_{p=0}^{Q_1-1} \sum_{q=0}^{Q_2-1} u_{pq} h_p(\xi_1) h_q(\xi_2),$$

$$u_{pq} = u^\delta(\xi_{1p}, \xi_{2q}),$$

$$\frac{\partial u^\delta}{\partial \xi_1}(\xi_1, \xi_2) = \sum_{p=0}^{P_1} \sum_{q=0}^{P_2} u_{pq} \frac{dh_p(\xi_1)}{d\xi_1} h_q(\xi_2).$$



$$\frac{\partial u^\delta}{\partial \xi_1}(\xi_{1i}, \xi_{2j}) = \sum_{p=0}^{P_1} \sum_{q=0}^{P_2} \left\{ u_{pq} \frac{dh_p(\xi_1)}{d\xi_1} \Big|_{\xi_{1i}} \delta_{qj} \right\} = \sum_{p=0}^{P_1} u_{pj} \frac{dh_p(\xi_1)}{d\xi_1} \Big|_{\xi_{1i}}$$

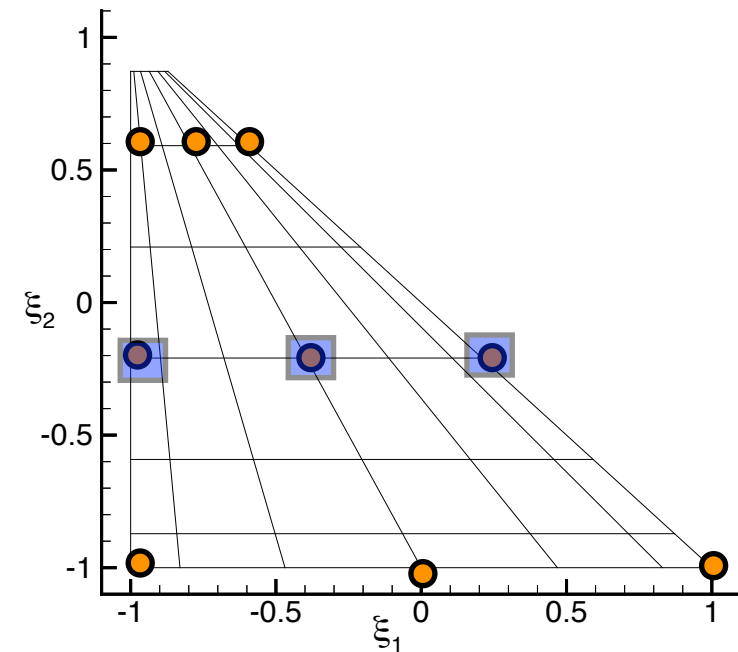
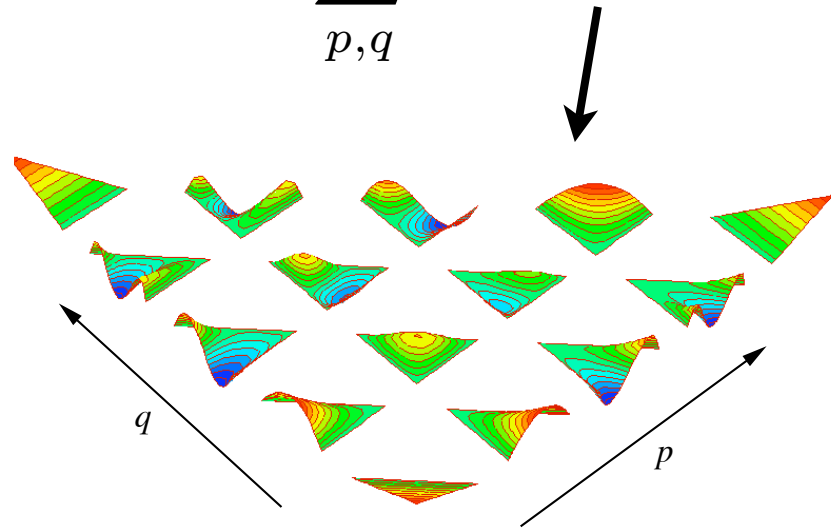
$$\frac{\partial u^\delta}{\partial \xi_2}(\xi_{1i}, \xi_{2j}) = \sum_{q=0}^{P_2} u_{iq} \frac{dh_q(\xi_2)}{d\xi_2} \Big|_{\xi_{2j}}.$$

Course Notes: Section 3.1.2.1

Numerical Differentiation: Standard Tri

StdRegions::StdTriExp::PhysDeriv()

$$u^\delta(\xi_1, \xi_2) = \sum_{p,q}^{P_1, P_2} \hat{u}_{pq} \phi_{pq}(\eta_1, \eta_2) \longleftrightarrow \sum_{p=0}^{P_1} \sum_{q=0}^{P_2} u_{pq} h_p(\eta_1) h_q(\eta_2), \quad u_{pq} = u^\delta(\eta_{1p}, \eta_{2q}),$$



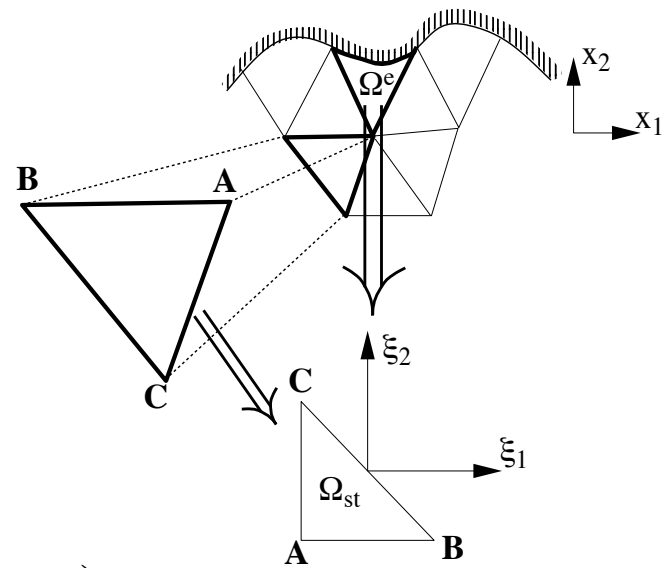
$$\eta_1 = \frac{2(1 + \xi_1)}{(1 - \xi_2)} - 1, \quad \eta_2 = \xi_2,$$

Apply chain rule:

$$\begin{pmatrix} \frac{\partial}{\partial \xi_1} \\ \frac{\partial}{\partial \xi_2} \end{pmatrix} = \begin{pmatrix} \frac{2}{(1 - \eta_2)} \frac{\partial}{\partial \eta_1} \\ 2 \frac{(1 + \eta_1)}{(1 - \eta_2)} \frac{\partial}{\partial \eta_1} + \frac{\partial}{\partial \eta_2} \end{pmatrix}$$

Numerical Differentiation: Local Regions

LocalRegions::QuadExp/TriExp::PhysDeriv()



$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial \xi_1}{\partial x_1} \frac{\partial}{\partial \xi_1} + \frac{\partial \xi_2}{\partial x_1} \frac{\partial}{\partial \xi_2} \\ \frac{\partial \xi_1}{\partial x_2} \frac{\partial}{\partial \xi_1} + \frac{\partial \xi_2}{\partial x_2} \frac{\partial}{\partial \xi_2} \end{bmatrix}$$

$$\chi_1(\xi_1, \xi_2)$$

$$\chi_2(\xi_1, \xi_2)$$

$$dx_1(\xi_1, \xi_2) = \frac{\partial x_1}{\partial \xi_1} d\xi_1 + \frac{\partial x_1}{\partial \xi_2} d\xi_2$$

a)

$$\begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} \end{bmatrix} \begin{bmatrix} d\xi_1 \\ d\xi_2 \end{bmatrix}$$

$$\begin{bmatrix} d\xi_1 \\ d\xi_2 \end{bmatrix} = \frac{1}{J} \begin{bmatrix} \frac{\partial x_2}{\partial \xi_2} & -\frac{\partial x_1}{\partial \xi_2} \\ -\frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_1} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix},$$

$$d\xi_1(x_1, x_2) = \frac{\partial \xi_1}{\partial x_1} dx_1 + \frac{\partial \xi_1}{\partial x_2} dx_2$$

$$\begin{bmatrix} d\xi_1 \\ d\xi_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial \xi_1}{\partial x_1} & \frac{\partial \xi_1}{\partial x_2} \\ \frac{\partial \xi_2}{\partial x_1} & \frac{\partial \xi_2}{\partial x_2} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}.$$

$$\underline{\frac{\partial \xi_1}{\partial x_1} = \frac{1}{J} \frac{\partial x_2}{\partial \xi_2}},$$

$$\underline{\frac{\partial \xi_1}{\partial x_2} = -\frac{1}{J} \frac{\partial x_1}{\partial \xi_2}},$$

$$\underline{\frac{\partial \xi_2}{\partial x_1} = -\frac{1}{J} \frac{\partial x_2}{\partial \xi_1}},$$

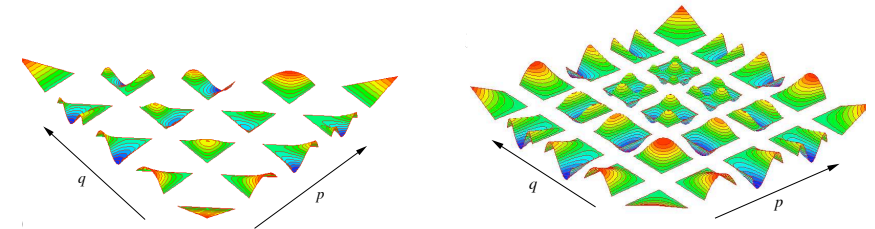
$$\underline{\frac{\partial \xi_2}{\partial x_2} = \frac{1}{J} \frac{\partial x_1}{\partial \xi_1}}.$$

Elemental forward transform

LocalRegions::QuadExp/TriExp::FwdTrans()

Expansion:

$$u^\delta = \sum_{p,q} \hat{u}_{pq} \phi_{pq}$$



Approximation:

$$u^\delta(\xi_1, \xi_2) - u(\xi_1, \xi_2) = R(u)$$

$$\left(\sum_{pq} \hat{u}_{pq} \phi_{pq}(\xi_1, \xi_2) \right) - u(\xi_1, \xi_2) = R(u).$$

Method of weighted residual:

$$(v, \sum_{p,q} \hat{u}_{pq} \phi_{pq}) - (v, u) = (v, \cancel{R(u)}),$$

$$(a, b) = \int_{\Omega^e} ab \, d\mathbf{x}$$

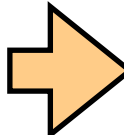
$$(v, \sum_{p,q} \hat{u}_{pq} \phi_{pq}) = (v, u).$$


Course Notes: Sections 1.1 & 3.1.5.3

Elemental forward transform

LocalRegions::QuadExp/TriExp::FwdTrans()

$$(v, \sum_{p,q} \hat{u}_{pq} \phi_{pq}) = (v, u).$$

Galerkin weight: $v(\xi_1, \xi_2) = \phi_{rs}(\xi_1, \xi_2)$.  $(\phi_{rs}, \sum_{p,q} \hat{u}_{pq} \phi_{pq}) = (\phi_{rs}, u)$

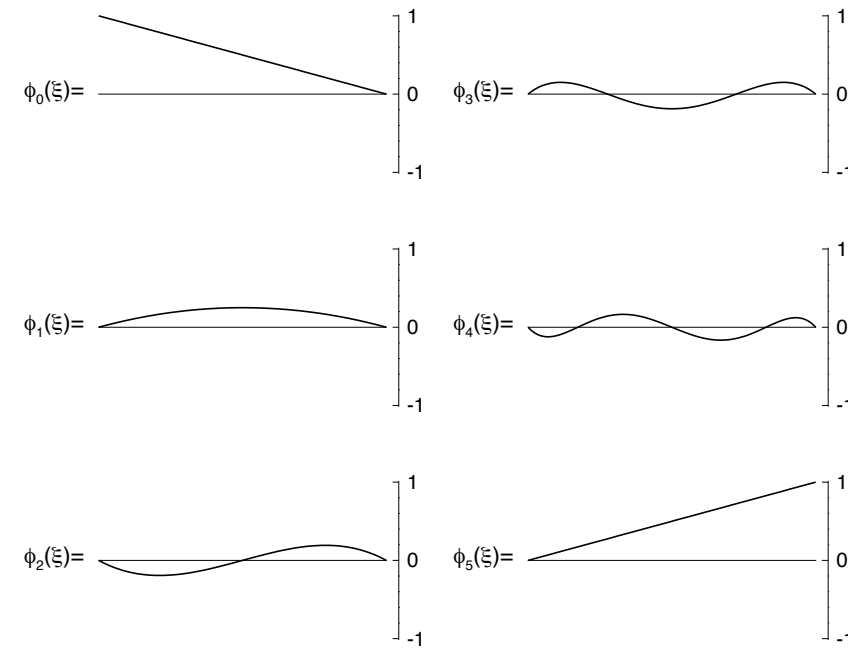
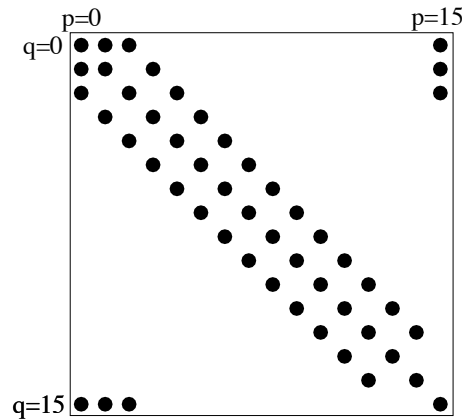

$$\sum_{p,q} (\phi_{rs}, \phi_{pq}) \hat{u}_{pq} = (\phi_{rs}, u),$$

$$\mathbf{M}[i][j] = \int_{\Omega^e} \phi_{i(r,s)}(x_1, x_2) \phi_{j(p,q)}(x_1, x_2) dx_1 dx_2 \quad \hat{\mathbf{u}}[i] = \hat{u}_{i(p,q)}$$
$$\mathbf{f}[i] = \int_{\Omega^e} \phi_{i(p,q)} u(x_1, x_2) dx_1 dx_2$$

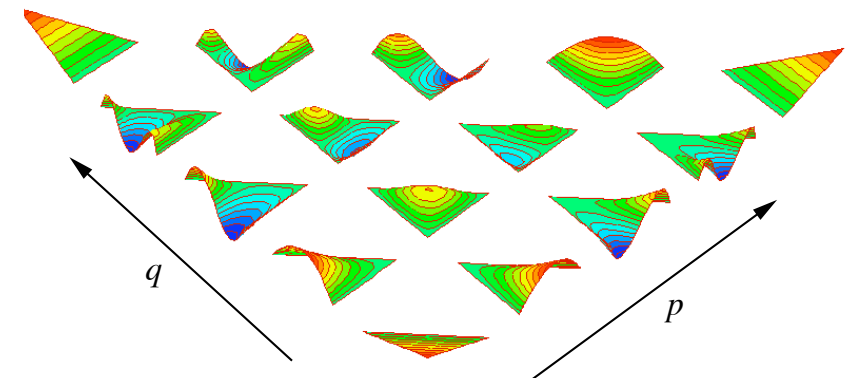
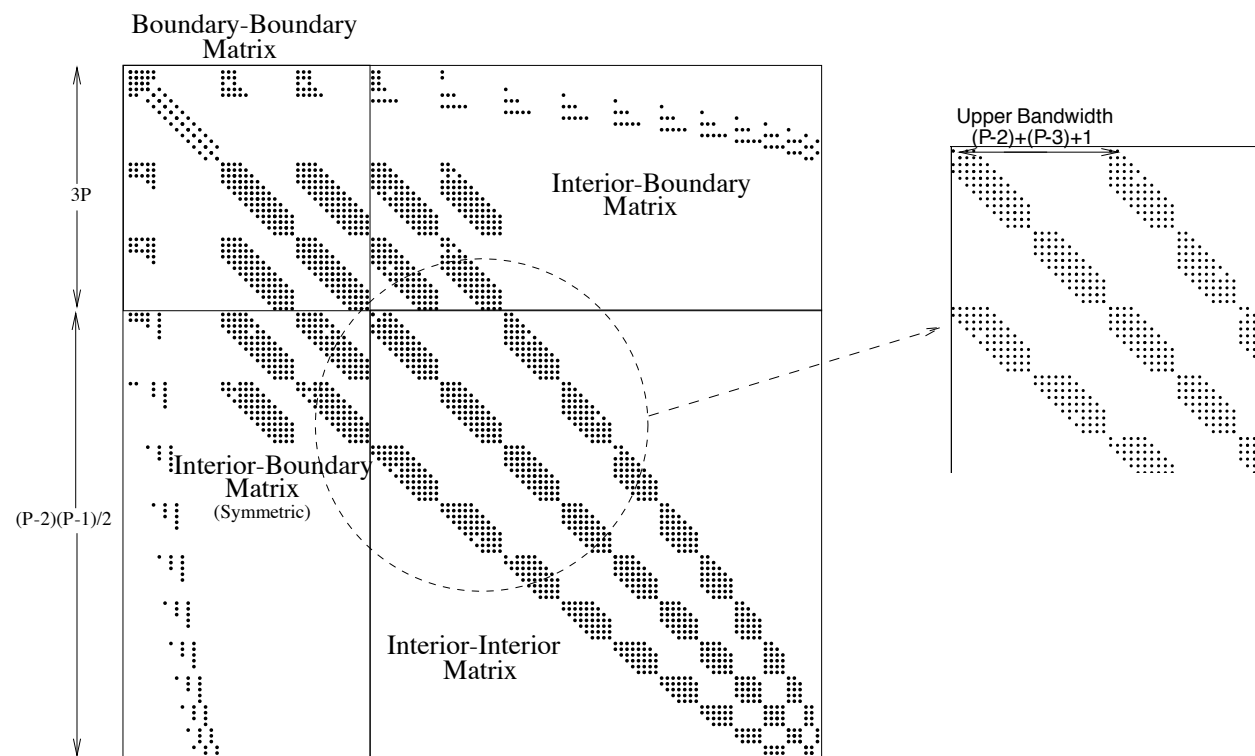
$$\hat{\mathbf{u}} = \mathbf{M}^{-1} \mathbf{f}$$

Elemental Mass matrix structure

Segment expansion $P = 15$

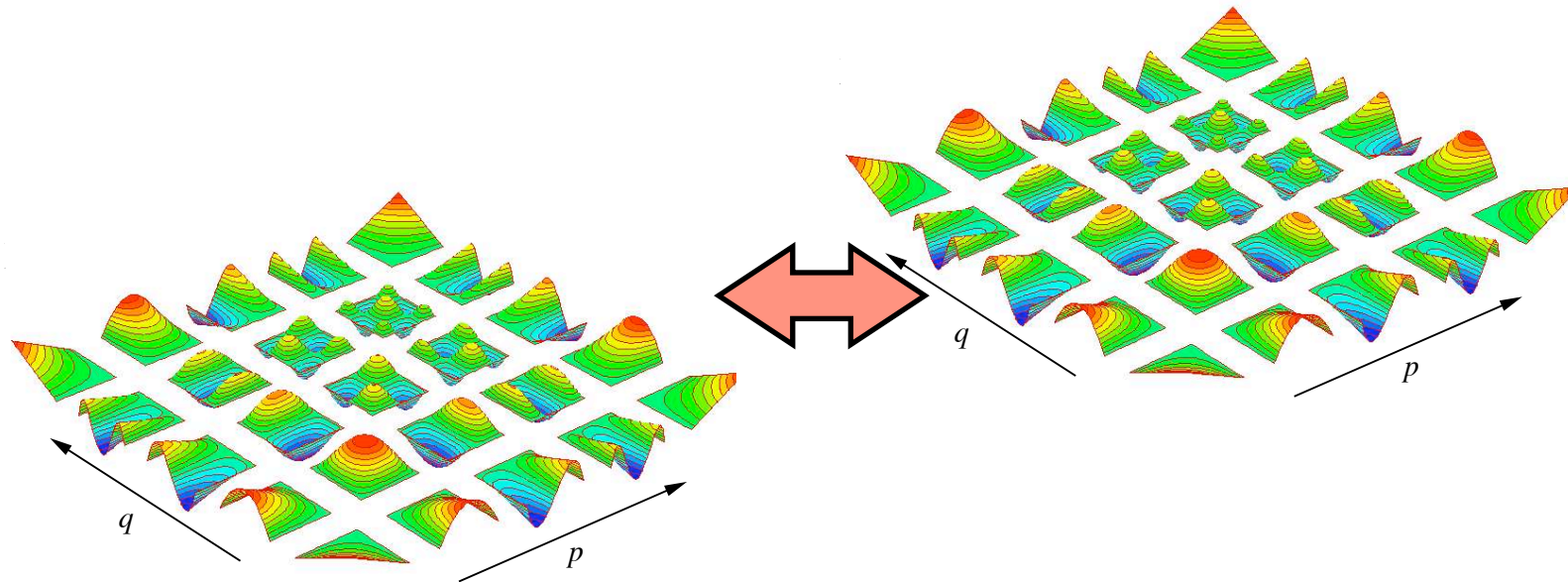


Triangular expansion $P=14$

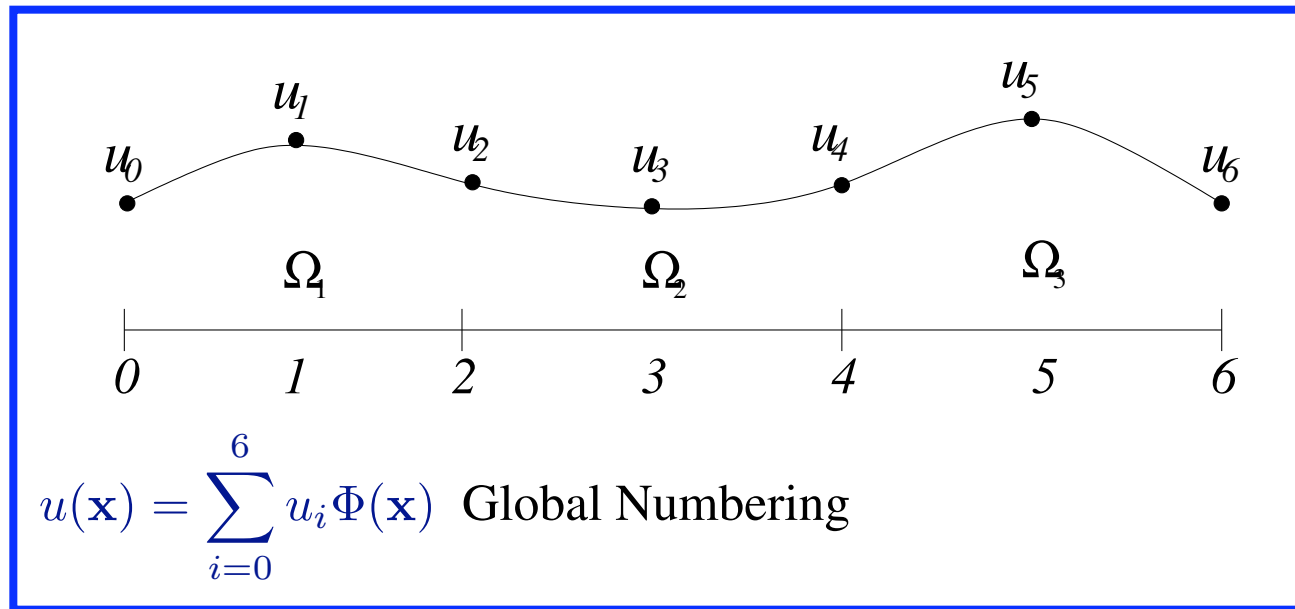


Global Assembly

- Classical continuous Galerkin FEM
- 2nd order PDE: $\nabla^2 u - \lambda u = -f$
- C_0 continuity sufficient.
- Use boundary interior decomposition to make continuous expansion.

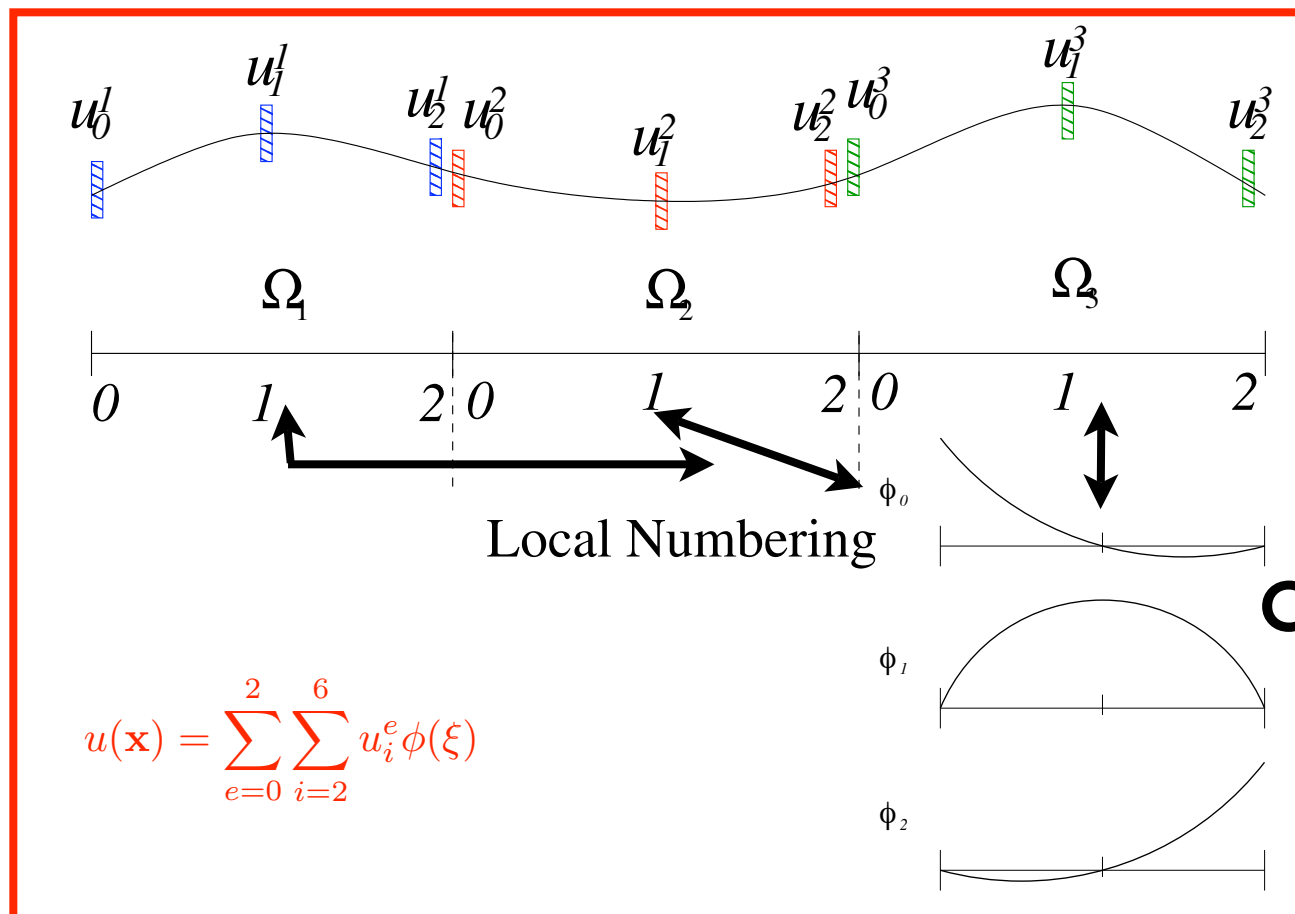


Global Assembly

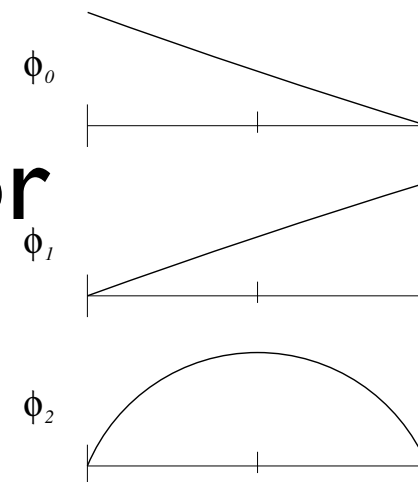


$$\tilde{\mathbf{u}}_l = \mathcal{A} \tilde{\mathbf{u}}_g$$

$$\tilde{\mathbf{u}}_l = \begin{bmatrix} u_0^1 \\ u_1^1 \\ u_2^1 \\ u_0^2 \\ u_1^2 \\ u_2^2 \\ u_0^3 \\ u_1^3 \\ u_2^3 \end{bmatrix} = \mathcal{A} \tilde{\mathbf{u}}_g = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u}_0 \\ \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \\ \tilde{u}_4 \\ \tilde{u}_5 \\ \tilde{u}_6 \end{bmatrix}$$



or

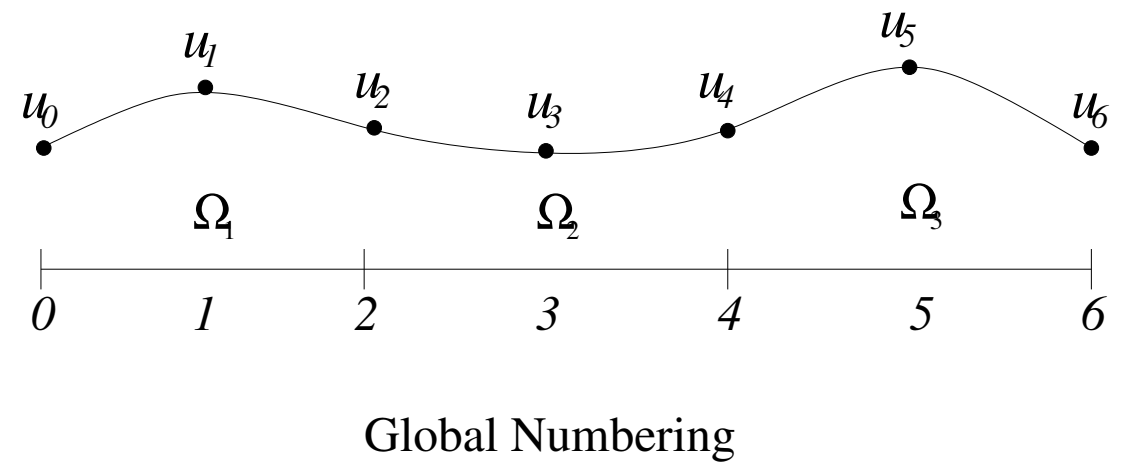


$$\tilde{\mathbf{i}}_g = \mathcal{A}^T \tilde{\mathbf{i}}_l$$

$$\mathcal{A}^T \mathcal{A} \neq \mathbf{I}$$

Global Assembly

$$\tilde{\mathbf{u}}_l = \begin{bmatrix} u_0^1 \\ u_1^1 \\ u_2^1 \\ u_0^2 \\ u_1^2 \\ u_2^2 \\ u_0^3 \\ u_1^3 \\ u_2^3 \end{bmatrix} = \mathcal{A} \tilde{\mathbf{u}}_g = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u}_0 \\ \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \\ \tilde{u}_4 \\ \tilde{u}_5 \\ \tilde{u}_6 \end{bmatrix}$$



$$\text{map}[1][p] = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \text{map}[2][p] = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \quad \text{map}[3][p] = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

```

for e = 1 : Nel
    for p = 0 : Pe
         $\tilde{\mathbf{u}}^e[p] = \tilde{\mathbf{u}}_g[\text{map}[e][p]]$ 
    end
end

for e = 1 : Nel
    for p = 0 : Pe
        for q = 0 : Pe
             $\mathbf{M}_g[\text{map}[e][p]][\text{map}[e][q]] =$ 
            end  $\mathbf{M}_g[\text{map}[e][p]][\text{map}[e][q]] + \mathbf{M}^e[p][q]$ 
        end
    end
end
```

Nektar++ code

