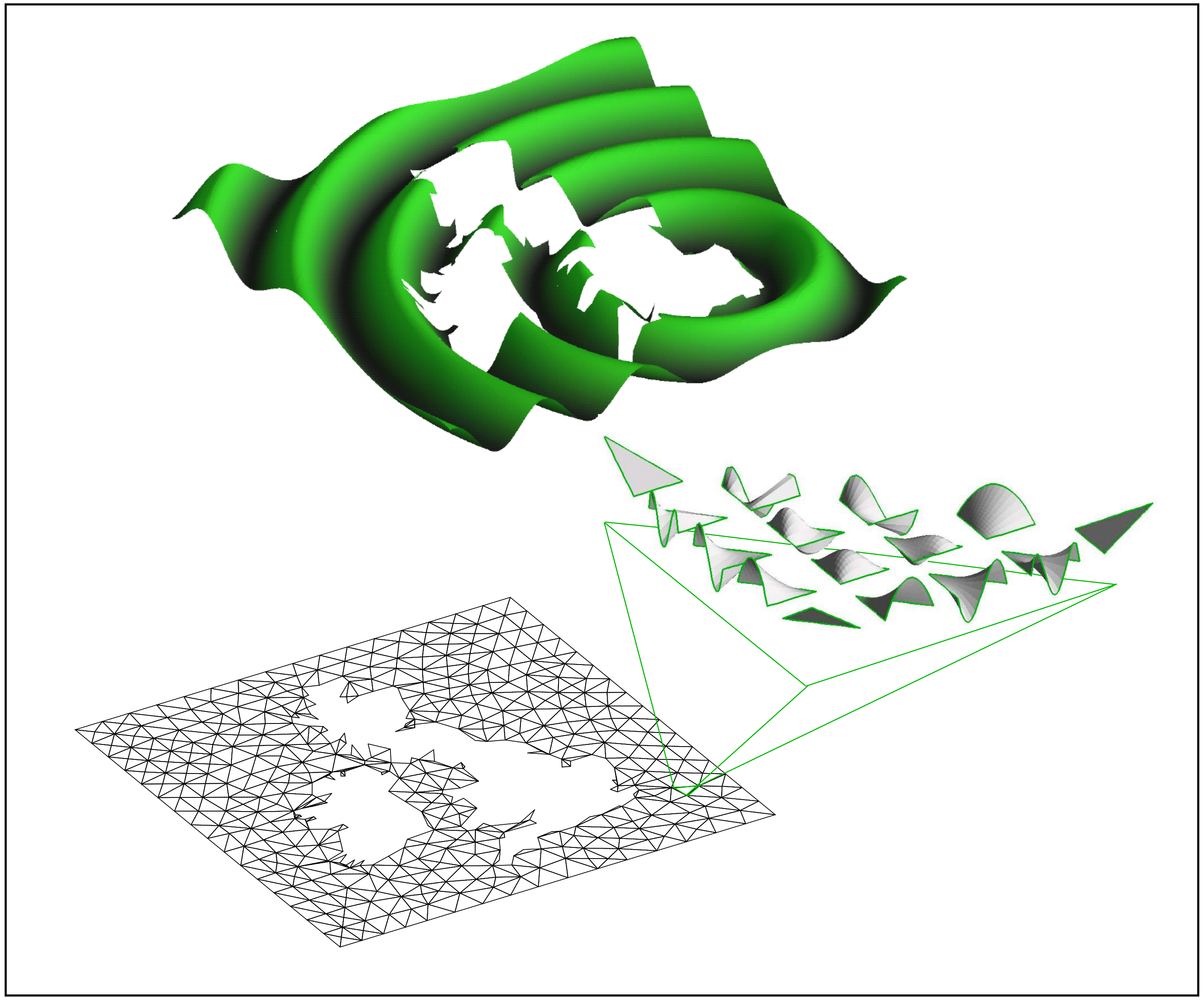
Expansions in Standard Regions



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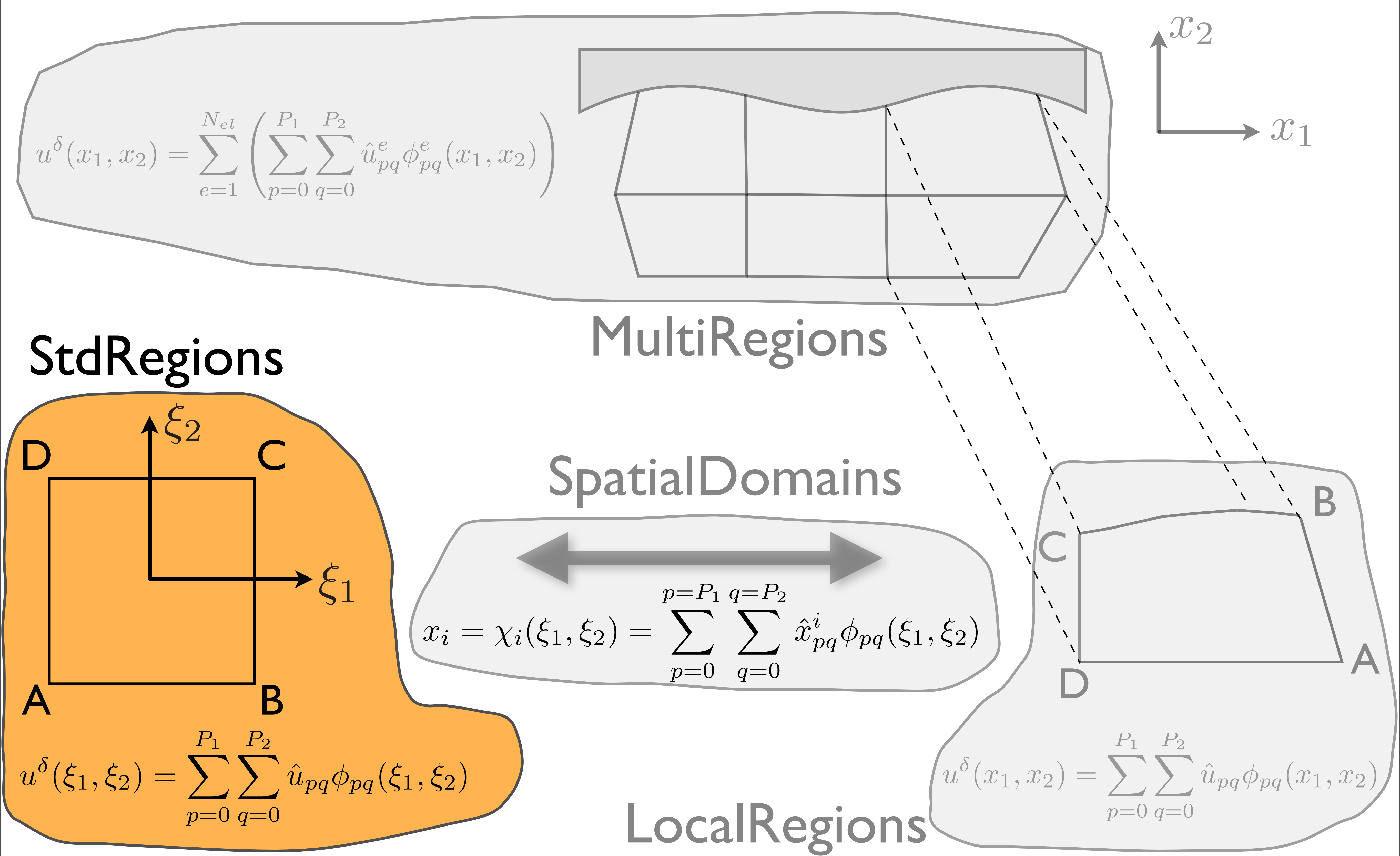
The big picture



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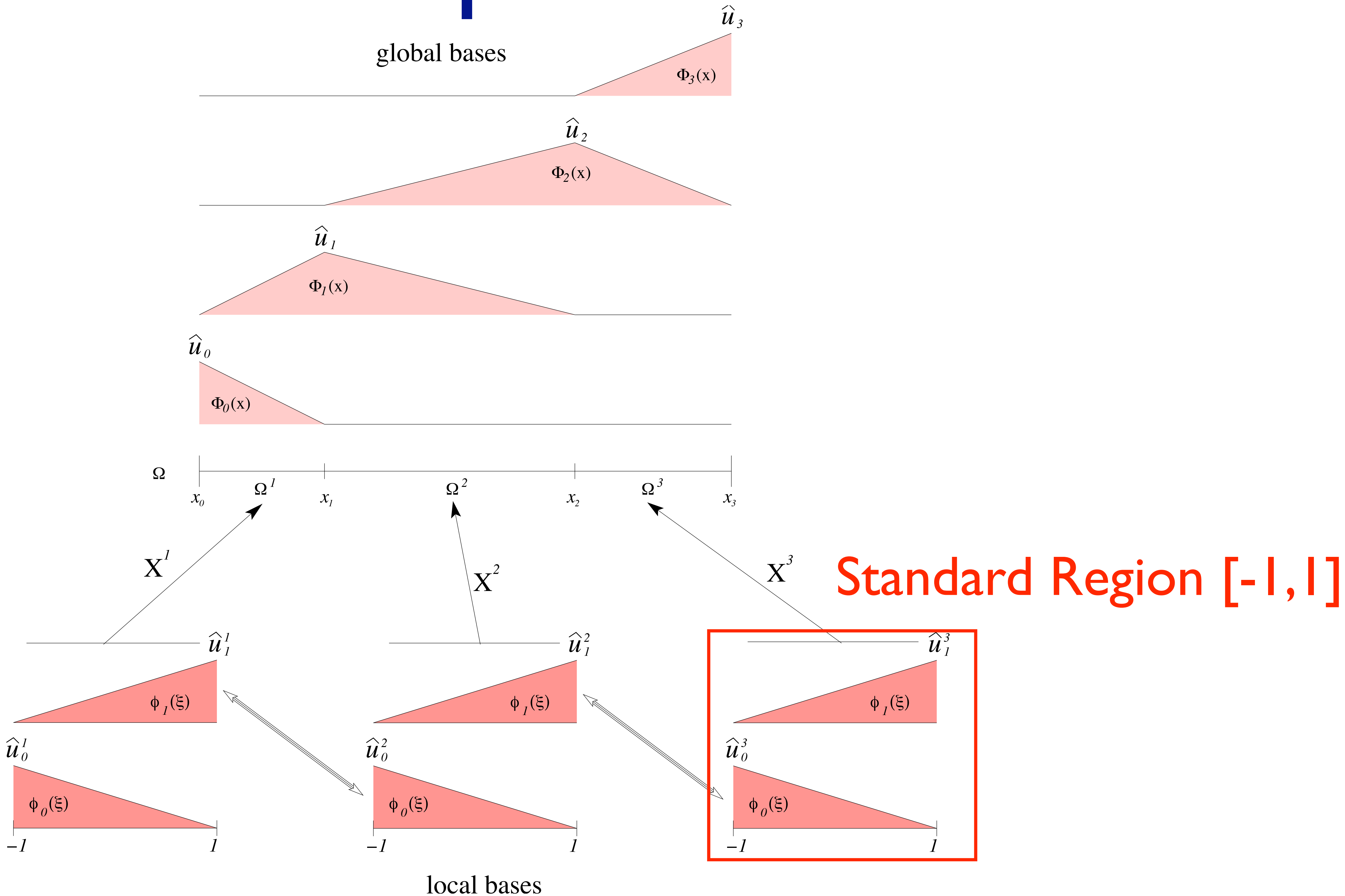
Outline

- Choice of Tensorial Expansions
  - Standard Segments (StdRegions::StdSegExp)
  - Standard Quadrilaterals (StdRegions::StdQuadExp)
  - Standard Triangles (StdRegions::StdTriExp)
- Sum Factorisation of tensorial bases (notes: 3.1.6)

ii

Classic linear finite elements expansion



Polynomial Expansion

1.3.2.1 Construction of a Polynomial Expansion

In an *hp* elemental discretisation we can apply a polynomial expansion of any order within each elemental region. It is therefore appropriate to start our discussion of p-type methods by considering what makes an acceptable p-type expansion in a single domain.

The steps involved in designing an elemental p-type expansion, which we will also later adopt in constructing the unstructured basis in section 2.2, are:

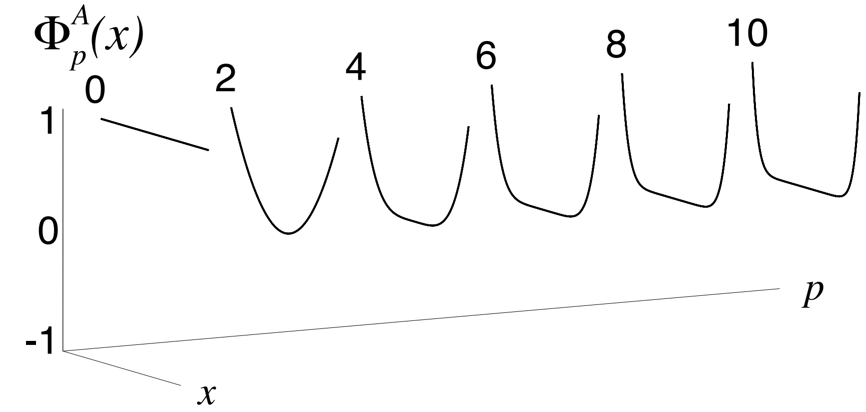
- Determine a favourable expansion within a standard region.
- Modify the expansion so that it can easily be numerically implemented.

In the first step, a favourable expansion is typically an orthogonal or near orthogonal set of functions within the standard regions. In the second step, the computational considerations of implementing this basis are taken into account and the basis is modified, if necessary, to facilitate this process. Typically, the basis is decomposed into contributions on the boundary and interior of the standard region since this simplifies the elemental decomposition process.

Choice of an Expansion Bases

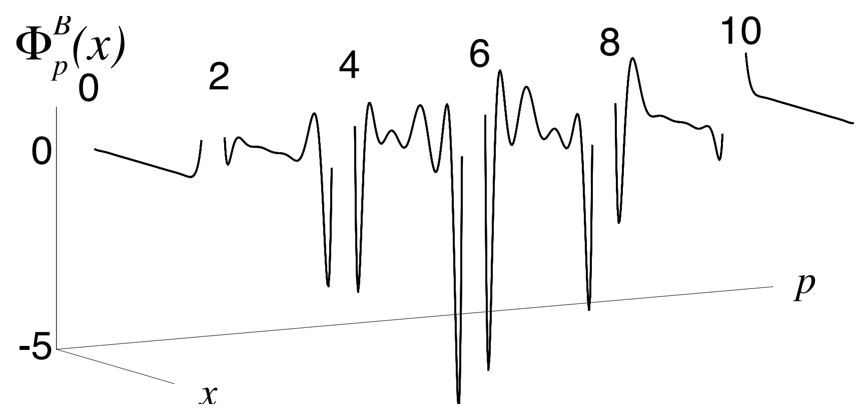
Consider three expansion:

1. • $\Phi_p^A(x)$, increases the order of x is a *moment* expansion (each order contributing an extra moment to the expansion).
   • Basis is *hierarchical modal.*



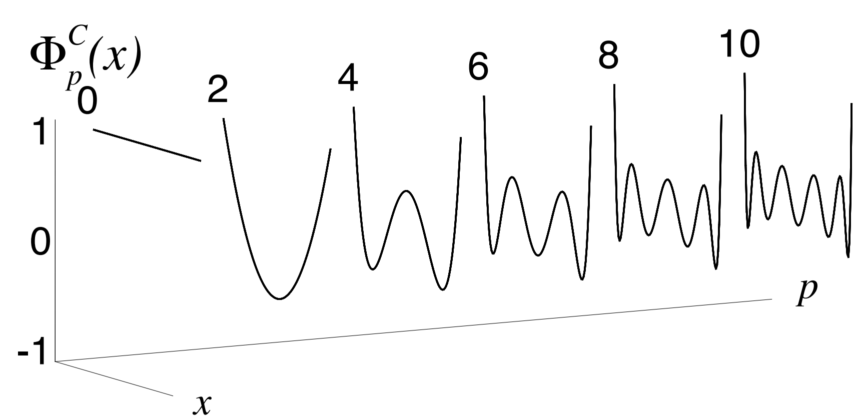
2. • $\Phi_p^B(x)$ is a Lagrange polynomial based on a series of $P+1$ nodal points x_q.
   • Lagrange polynomial is a non-hierarchical basis
   • The Lagrange basis has the property that $\Phi_p^B(x_q) = \delta_{pq}$

$$u^\delta(x) = \sum_{p=0}^{P} \hat{u}_p \Phi_p^B(x),$$



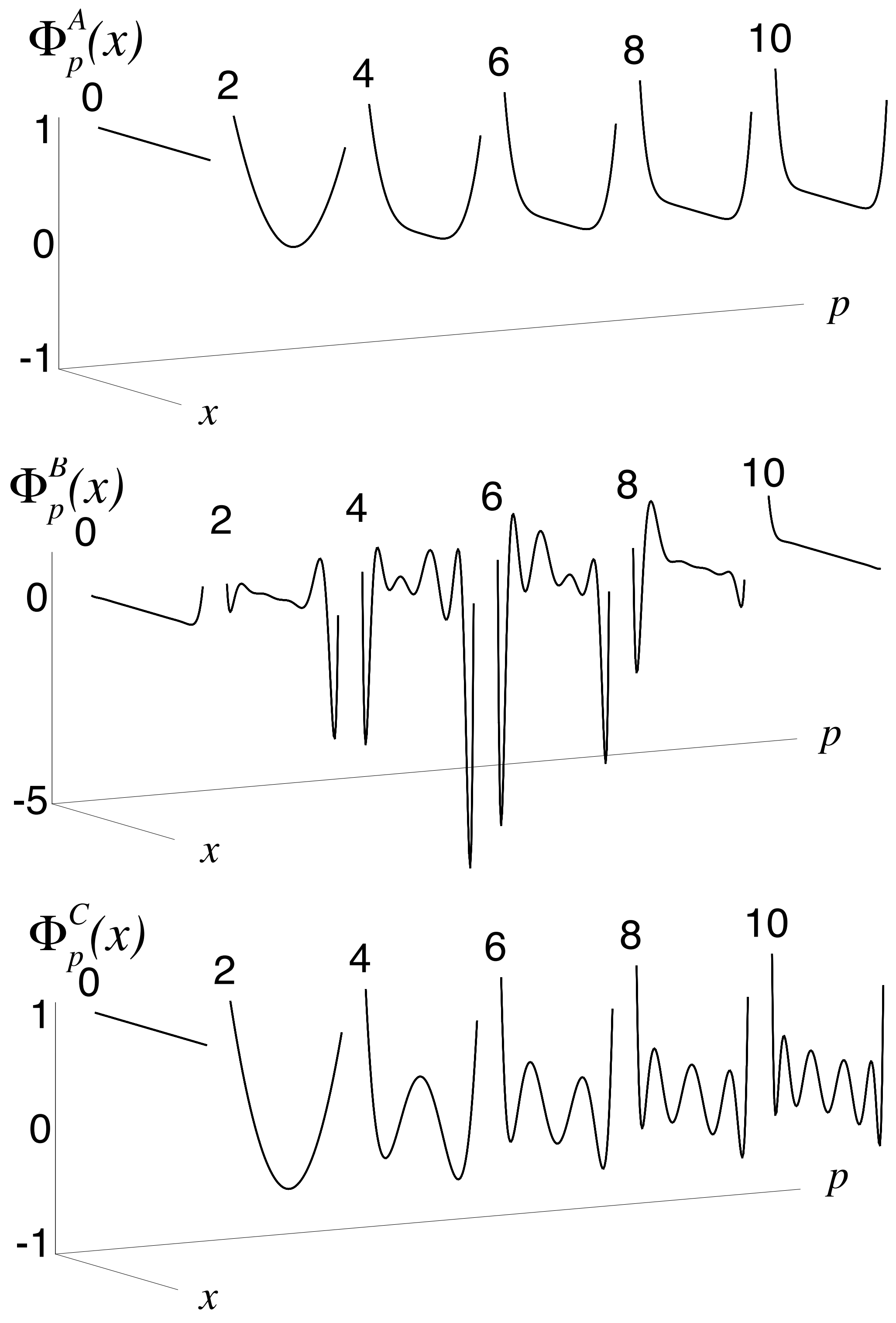
3. $\Phi_p^c(x)$, is a hierarchical modal expansion.
   • Based on the Legendre polynomial $L_p(x)$. which is orthogonal in the Legendre inner product

$$(L_p(x), L_q(x)) = \int_{-1}^{1} L_p(x) L_q(x) dx = \left(\frac{2}{2p+1}\right) \delta_{pq}.$$

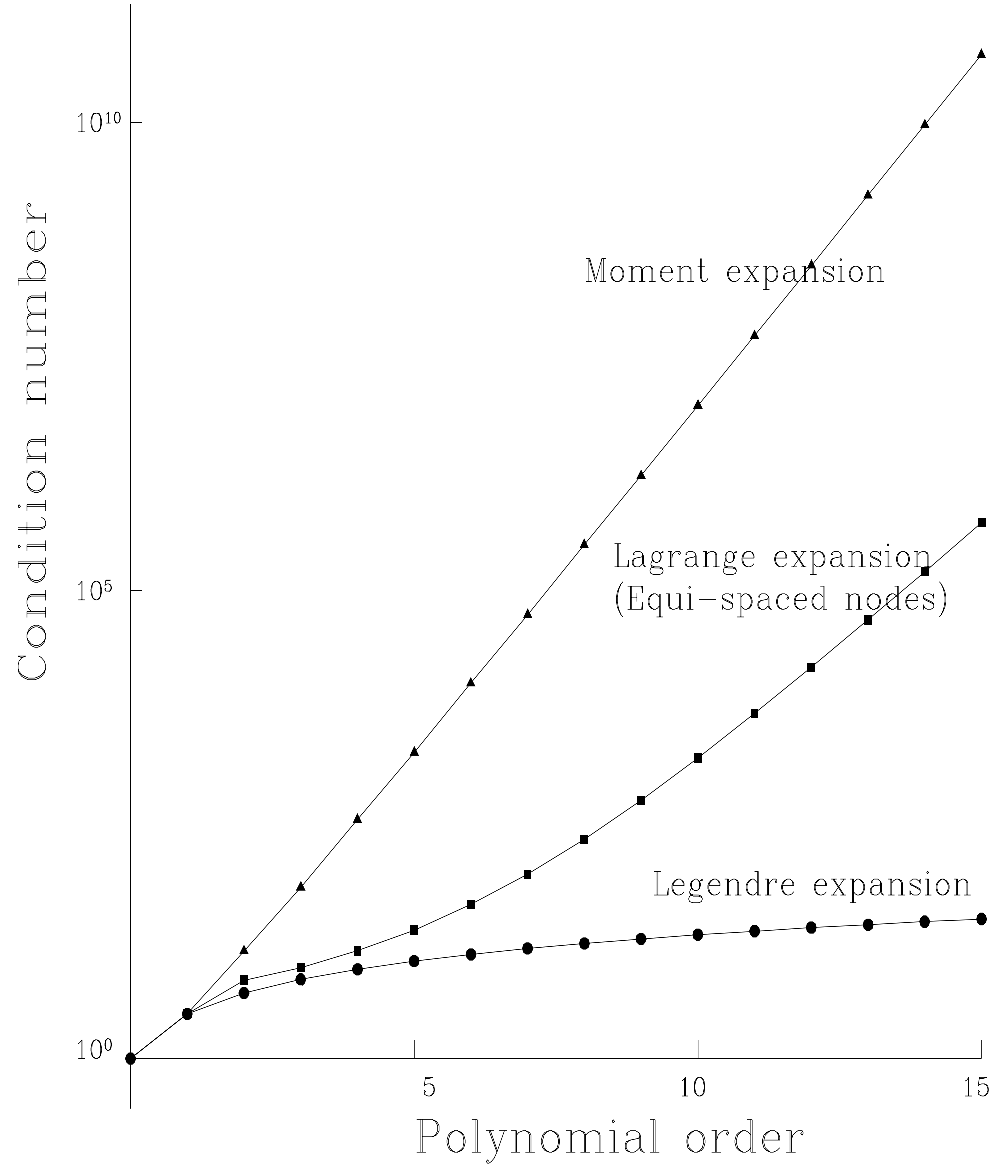


Notes section: 1.3.2.1

Choice of an Expansion Bases



$$\kappa_2 = ||\mathbf{M}||_2||\mathbf{M}^{-1}||_2$$



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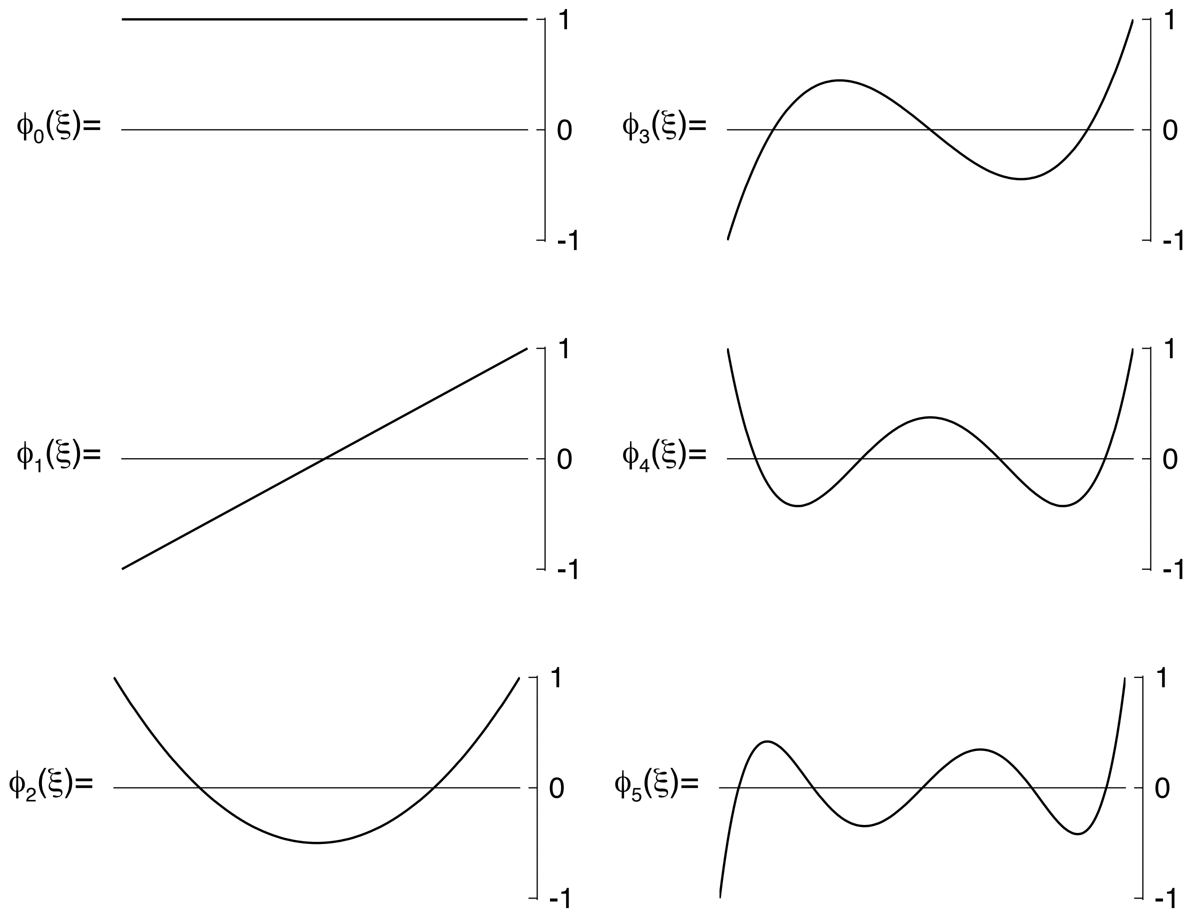
Notes section: 1.3.2.1

Legendre expansion



$$\Omega_{st} = \{\xi | -1 \leq \xi \leq 1\} \ ,$$

$$\phi_p(\xi) \mapsto L_p(\xi) \equiv P_p^{(0,0)}(\xi) \ , \quad 0 \leq p \leq P \ .$$

Jacobi Polynomials

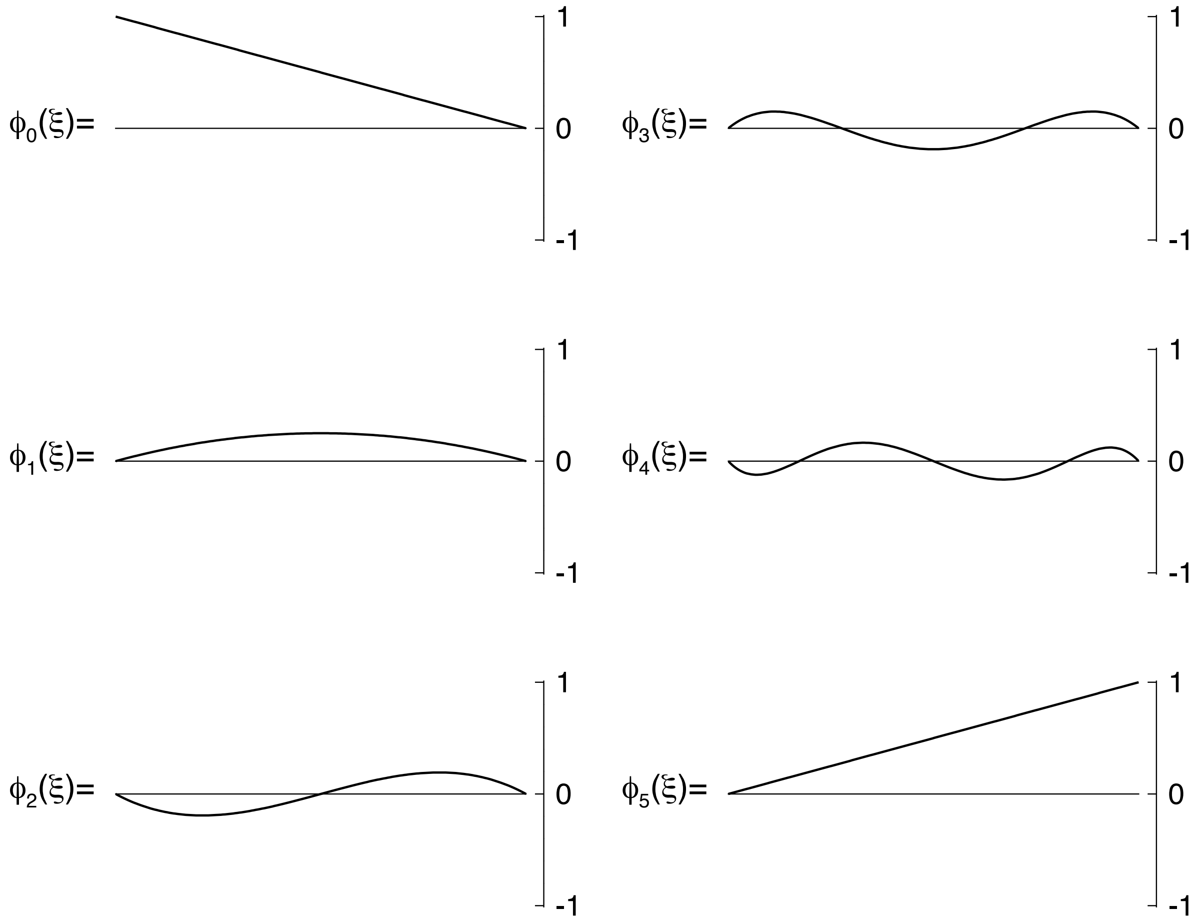
$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_n^{\alpha,\beta}(x) P_i^{\alpha,\beta}(x) \, d\xi = C\delta_{ni} \ ,$$

Notes section: 1.3.2.1

Boundary-Interior Decomposition

- "Best" choice for our expansion appears to be the Legendre polynomial.
- But also want to combine the expansion with the h-type elemental decomposition.
- Difficulty arises when we try to ensure a degree of continuity in the global expansion at elemental boundaries.
- Numerically efficient way of achieving C^0 continuity is to design an expansion where only some modes have a magnitude at elemental boundary
- This type of decomposition is known as *boundary* and *interior* decomposition.

P-type finite elements

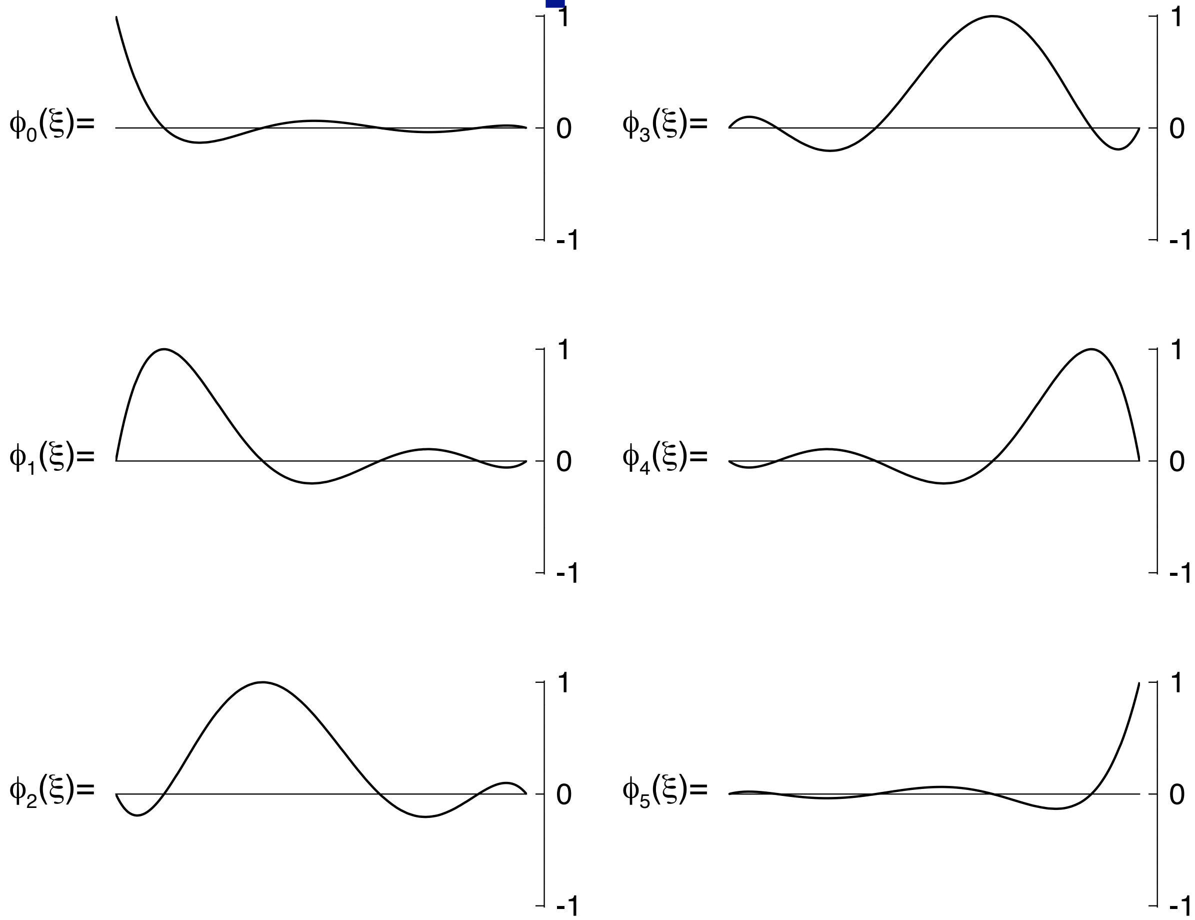


$$\Omega_{st} = \{\xi | -1 \leq \xi \leq 1\} \ ,$$

$$\phi_p(\xi) \mapsto \psi_p(\xi) = \begin{cases} \left(\frac{1-\xi}{2}\right) & p = 0 \\ \left(\frac{1-\xi}{2}\right)\left(\frac{1+\xi}{2}\right) P_{p-1}^{1,1}(\xi) & 0 < p < P \\ \left(\frac{1+\xi}{2}\right) & p = P \end{cases}$$

Notes section: 1.3.3.3

Spectral Elements



$$\Omega_{st} = \{\xi | -1 \leq \xi \leq 1\} \ ,$$

$$h_p(x) = \frac{\prod_{q=0,q\neq p}^{Q-1}(x - x_q)}{\prod_{q=0,q\neq p}^{Q-1}(x_p - x_q)} \ .$$

$$\phi_p(\xi) \mapsto h_p^{gl}(\xi) = \begin{cases} 1\,, & \xi = \xi_p\,, \\ \dfrac{(\xi-1)(\xi+1)\dfrac{\partial L_P(\xi)}{\partial \xi}}{P(P+1)L_P(\xi_p)(\xi_p - \xi)}\,, & \text{otherwise}\,, \end{cases} \qquad 0 \leq p \leq P\,. \qquad (20)$$

Collocation property: $u_\delta(\xi_q) = \sum_{p=0}^{P} \tilde{u}_p h_p(\xi_q) = \sum_{p=0}^{P} \tilde{u}_p \delta_{pq} = \tilde{u}_q.$

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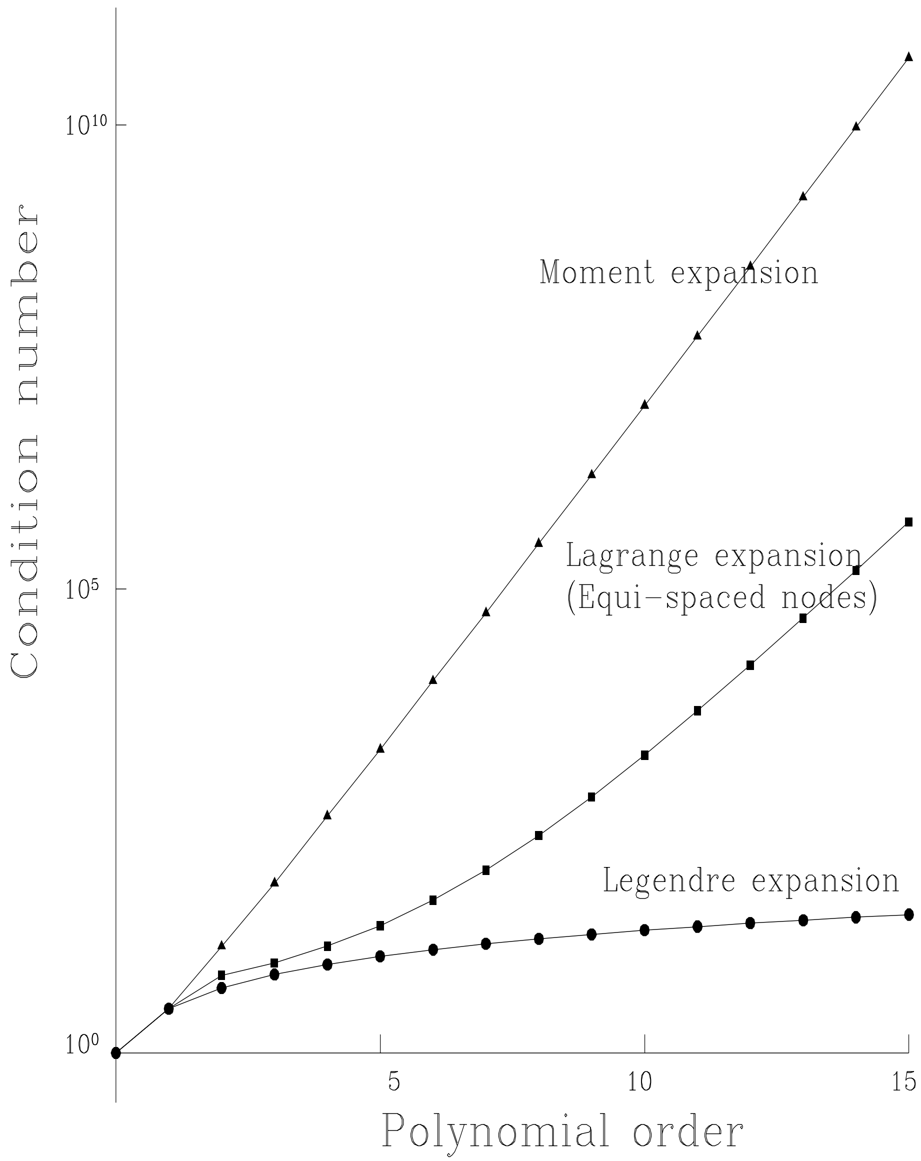
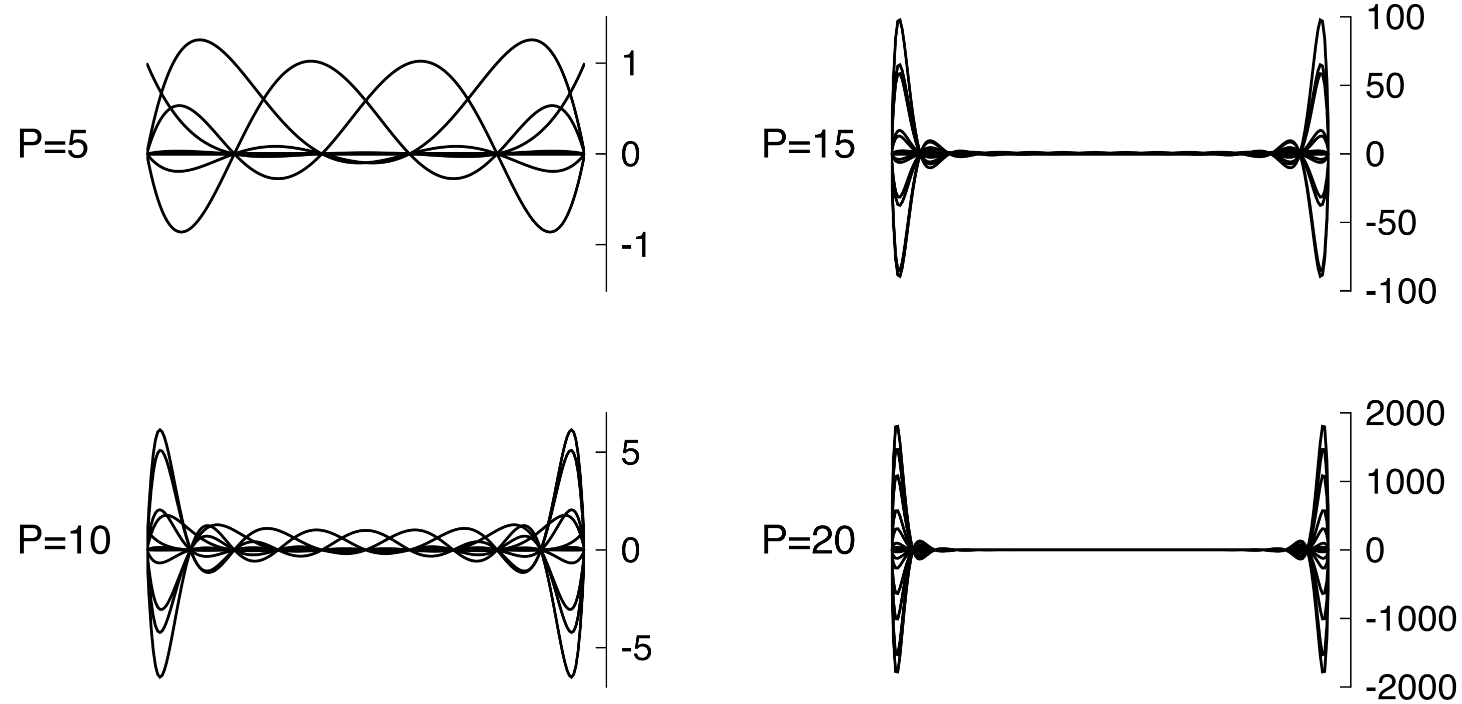
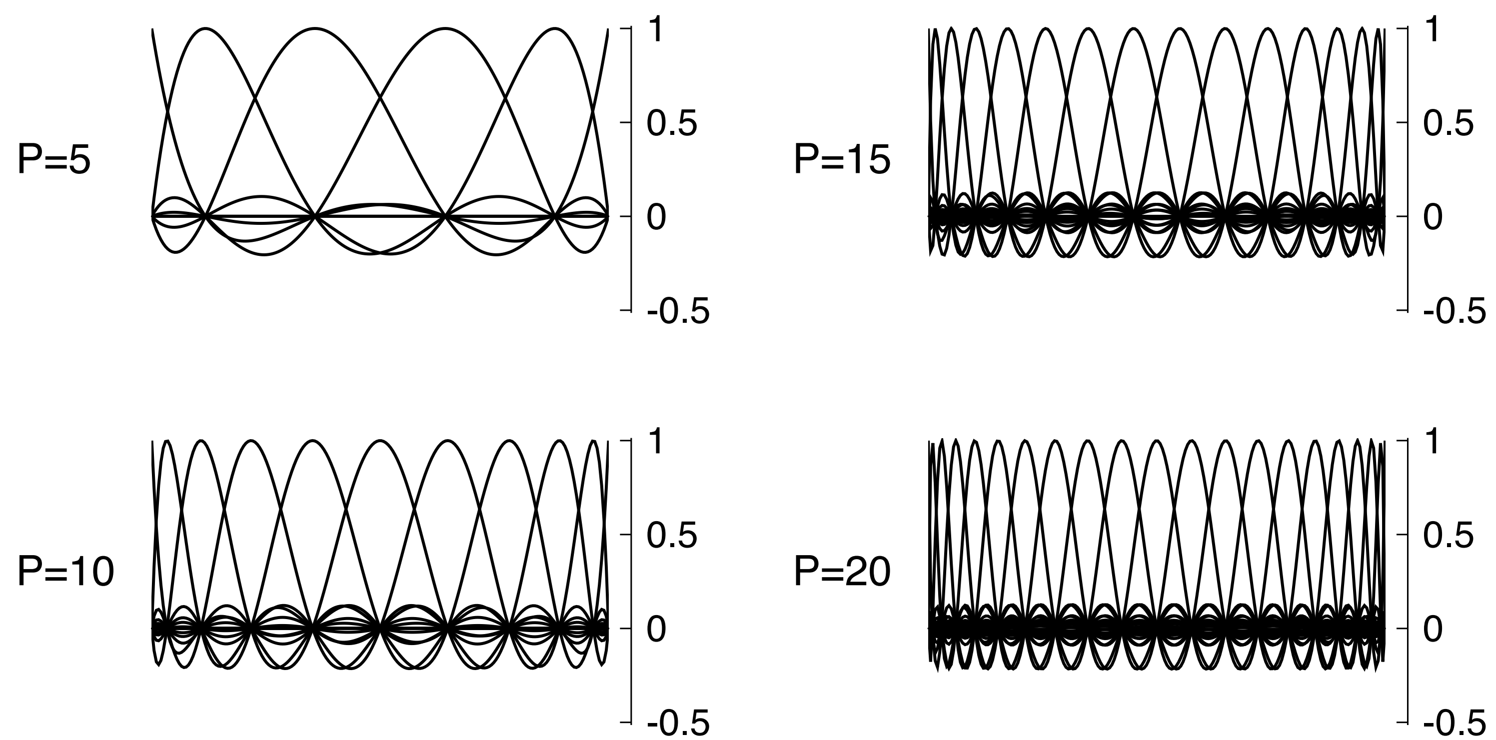
Notes section: 1.3.4.2

Spectral element Conditioning:





Lagrange nodal expansions through the equi-spaced points for polynomial orders of $P = 5, 10, 15$ and 20.



Lagrange nodal expansions through the Gauss-Lobatto points for polynomial orders of $P = 5, 10, 15$ and 20.

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Outline

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ii

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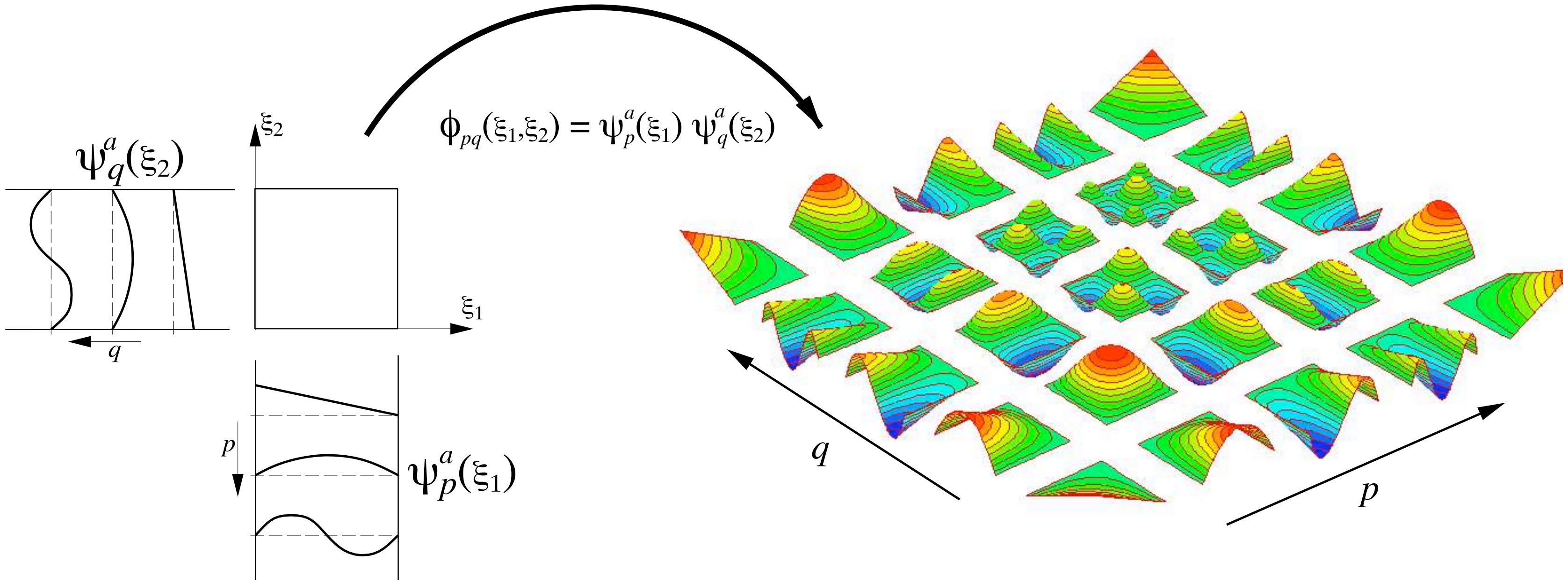
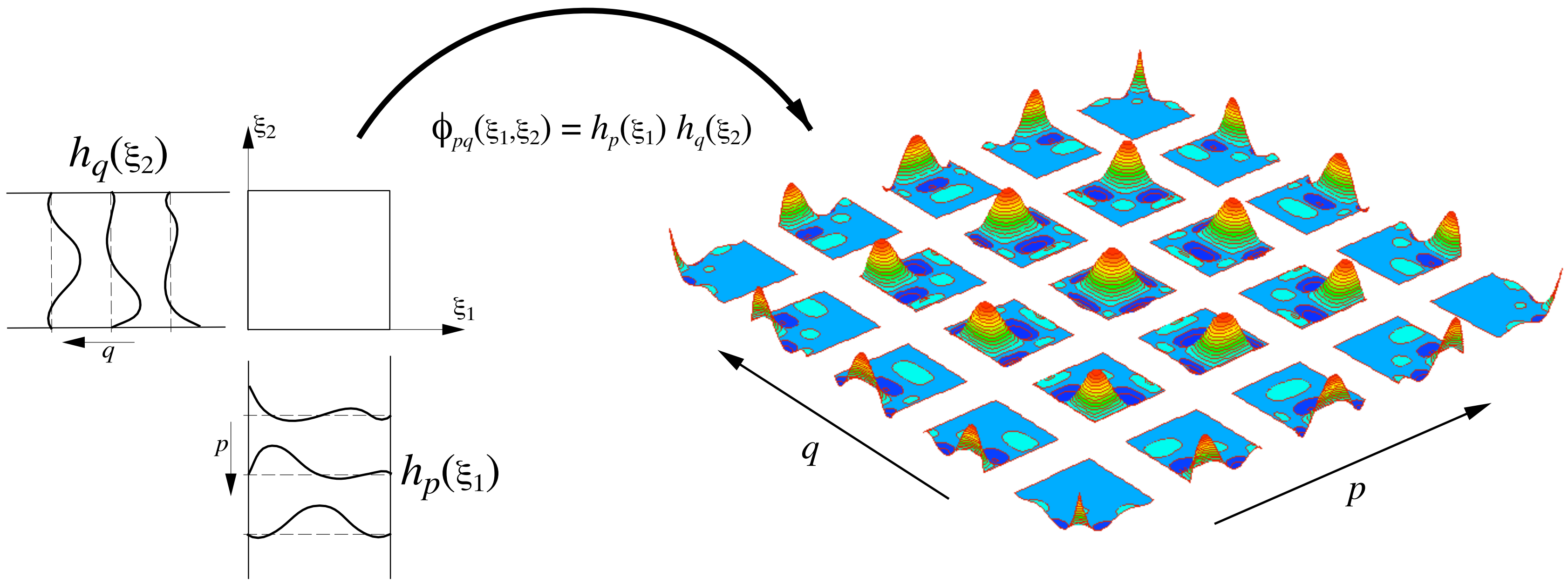
Spectral element/*P*-type finite element



p-type finite element- hierarchical basis



Spectral element - collocation basis

Notes section: 2.1.1

Outline

- Choice of Tensorial Expansions
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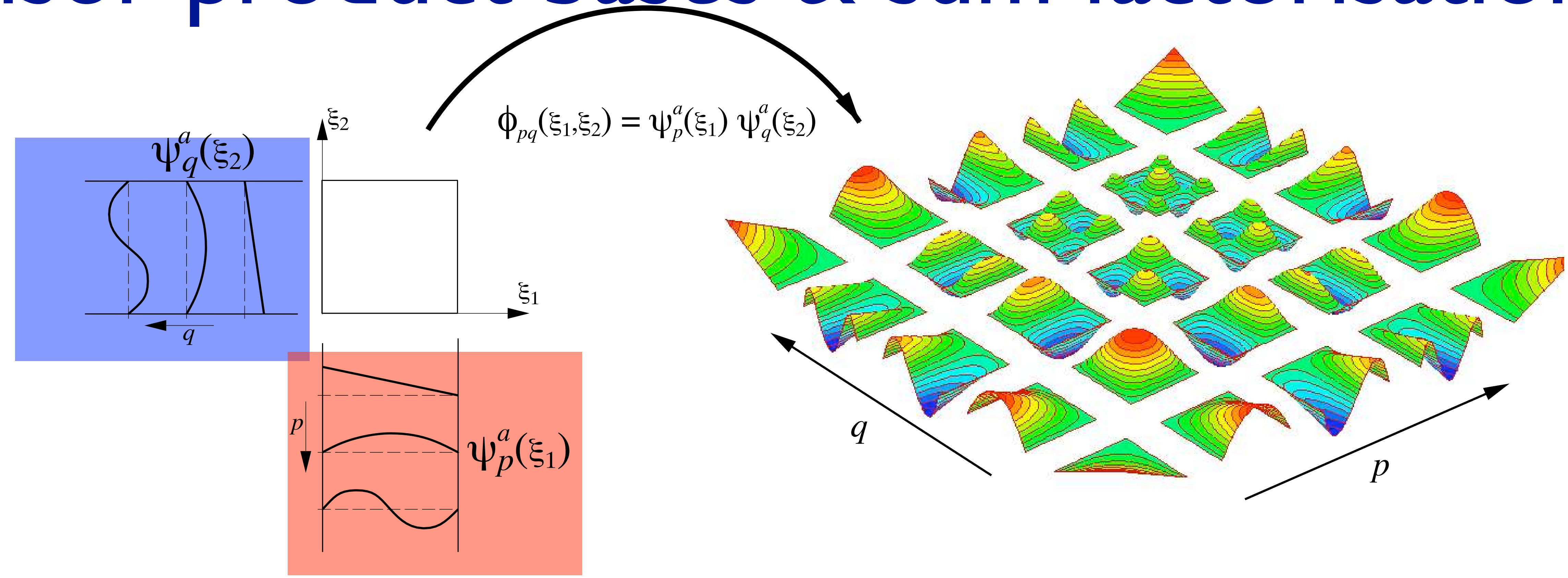
- Sum Factorisation of tensorial bases (notes: 3.1.6)

ii

Tensor product bases & sum factorisation



Inner product:

$$I_{pq} = \int_{\Omega^e} \phi_{pq}(\xi_1, \xi_2) u(\xi_1, \xi_2) = \sum_i \sum_j \phi_{pq}(\xi_{1,i}, \xi_{2,j}) u(\xi_{1,i}, \xi_{2,j})$$

$$I_{pq} = \sum_i^P \sum_j^P \psi^a_p(\xi_{1,i}) \psi^a_q(\xi_{2,j}) u(\xi_{1,i}, \xi_{2,j}) \sim O(P^4)$$

$$I_{pq} = \sum_i^P \psi^a_p(\xi_{1,i}) \; f_q(\xi_2, i) \qquad\qquad \sim O(P^3)$$

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Notes section: 3.1.6.2

Outline

- Choice of Tensorial Expansions
  - Standard Segments (StdRegions::StdSegExp)
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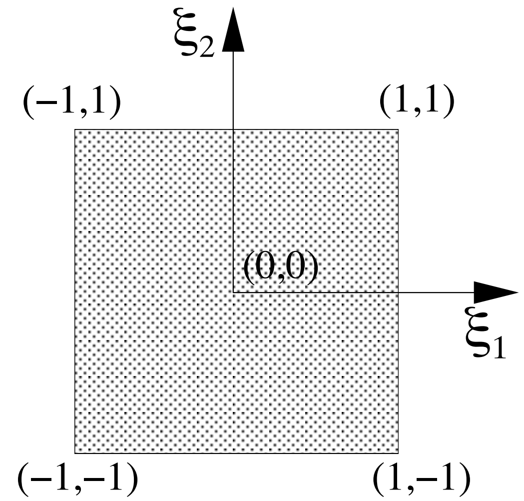
  - Standard Triangles (StdRegions::StdTriExp)
- Sum Factorisation of tensorial bases (notes: 3.1.6)

ii

Unstructured coordinate system

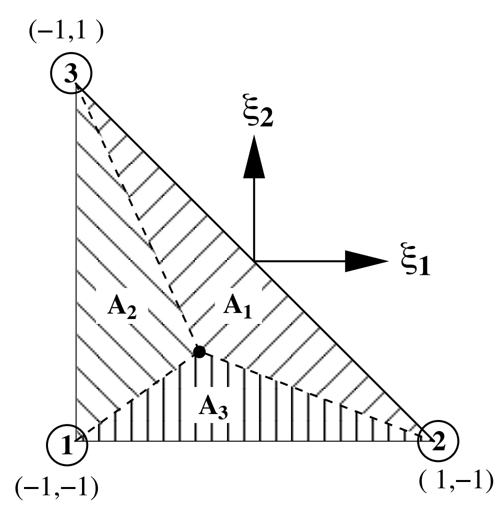
Quadrilateral system



$$\left.\begin{matrix}\xi_1 \\ \xi_2\end{matrix}\right\}$$

2 coordinates in 2 dimensions (rotationally symmetric)

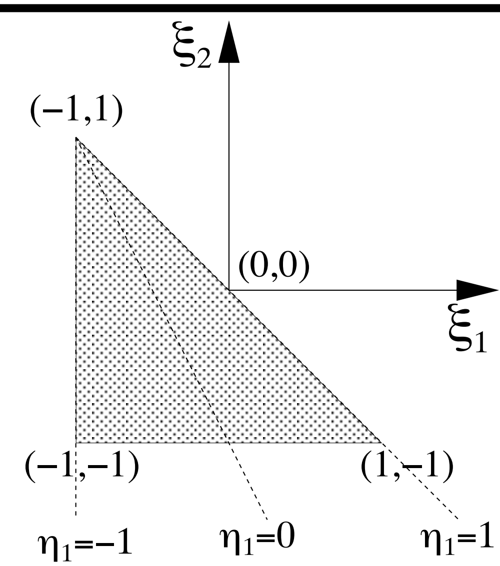
Barycentric/Area system



$$\left.\begin{matrix} l_1 = \dfrac{A_1}{A} \\ l_2 = \dfrac{A_2}{A} \\ l_3 = \dfrac{A_3}{A} \end{matrix}\right\}$$

3 coordinates in 2 dimensions (rotationally symmetric)

Collapsed cordinates

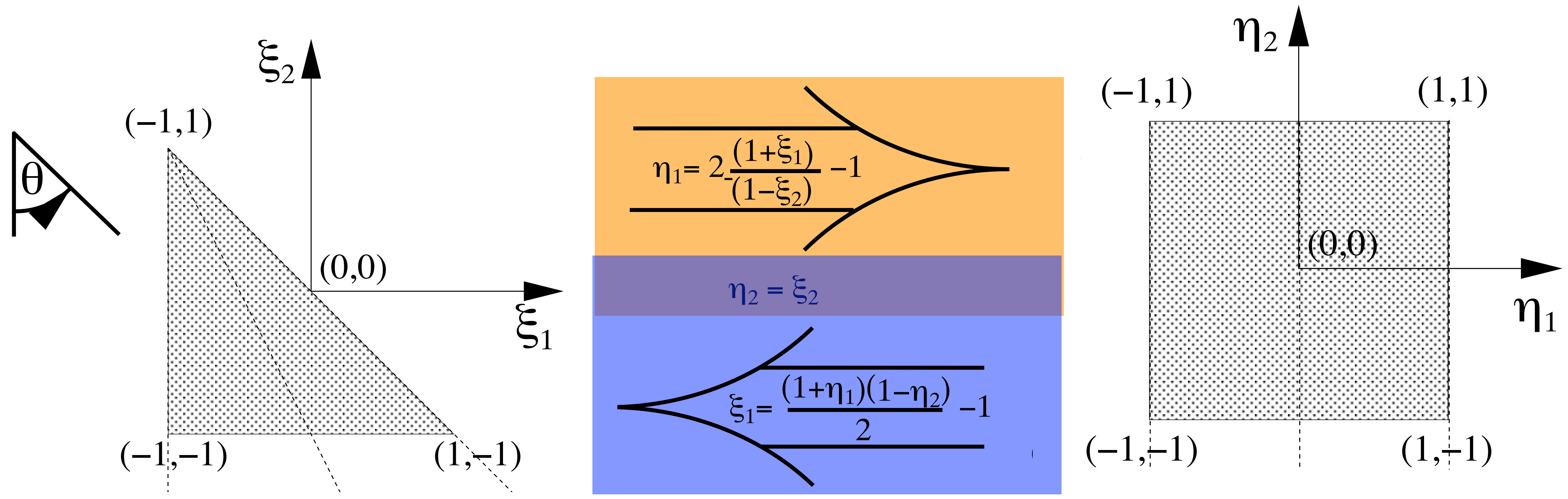


$$\left.\begin{matrix}\eta_1 \\ \eta_2\end{matrix}\right\}$$

2 coordinates in 2 dimensions (not rotationally symmetric)

Notes section: ??

Collapsed coordinate system



$$\mathcal{T}_{st} = \{(\xi_1\,,\xi_2)|-1 \leq \xi_1\,,\xi_2\,;\xi_1+\xi_2 \leq 0\}$$

$$\mathcal{T}_{st} = \{(\eta_1\,,\eta_2)|-1 \leq \eta_1\,,\eta_2 \leq 1\}\;,$$

$$\boxed{\eta_1 = 2\frac{(1+\xi_1)}{(1-\xi_2)} - 1\,, \qquad \eta_2 = \xi_2\,,}$$

$$\boxed{\xi_1 = \frac{(1+\eta_1)(1-\eta_2)}{2} - 1\,, \qquad \xi_2 = \eta_2}.$$

Analogous system used in cylindrical coordinate system

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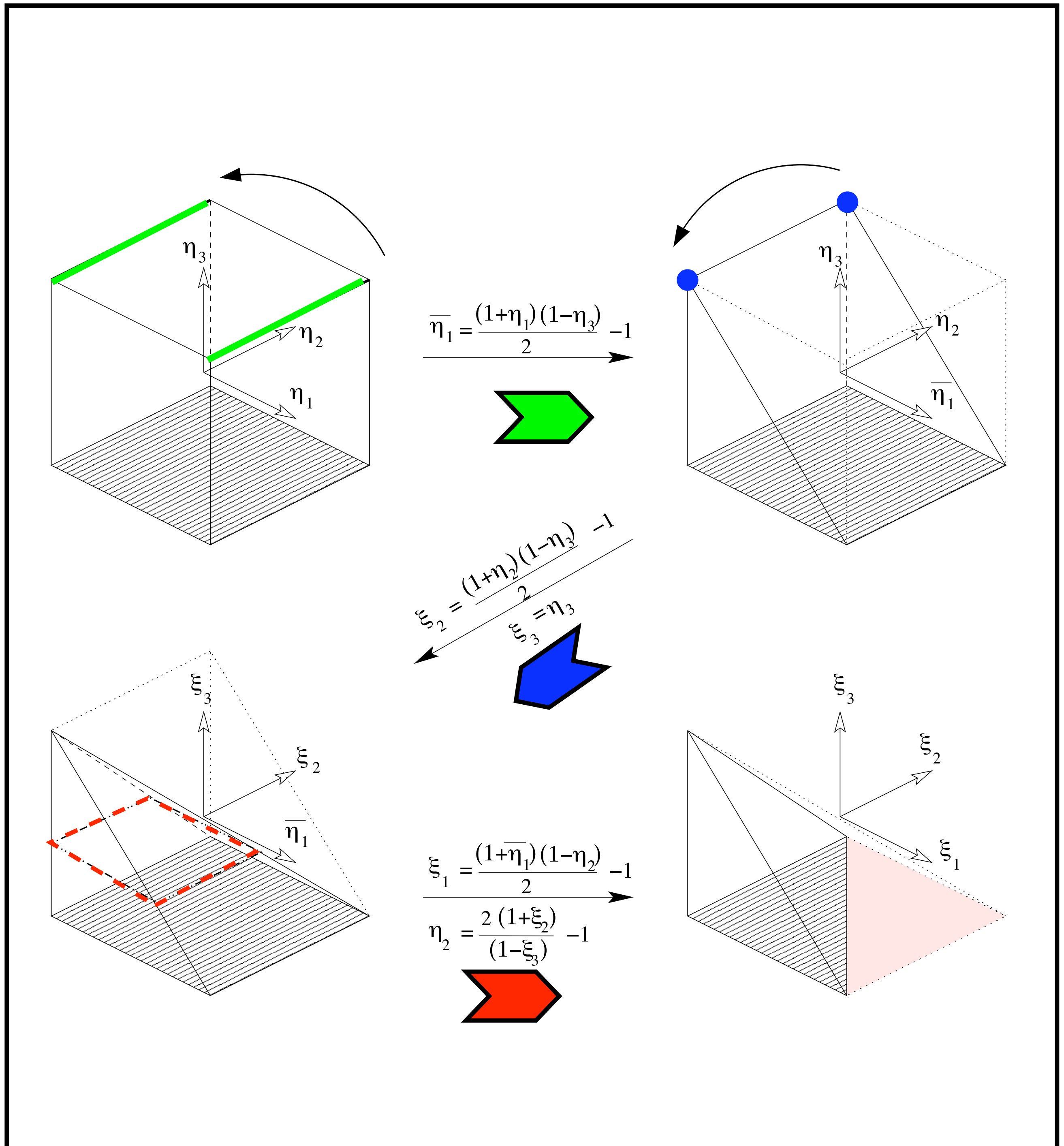
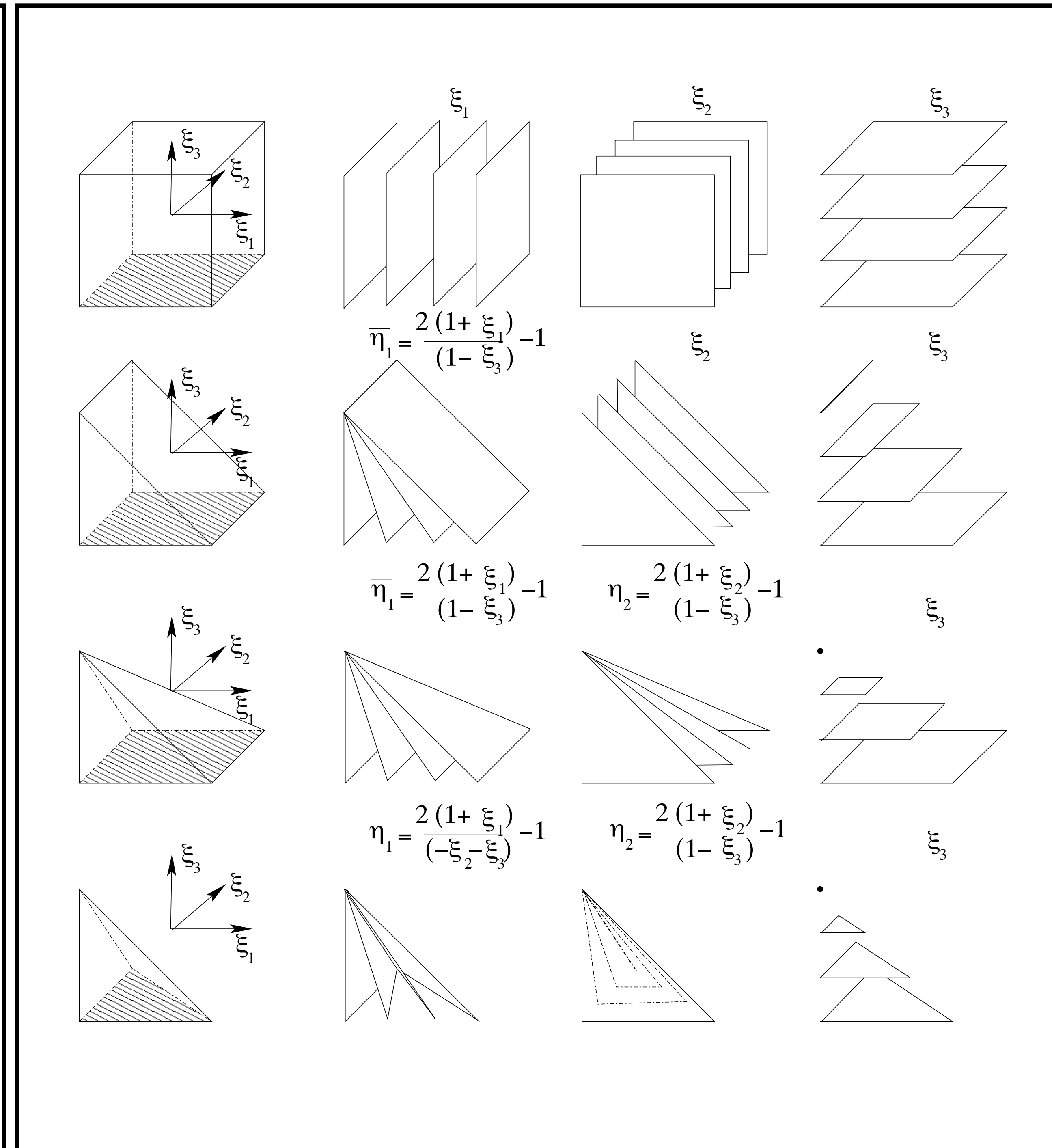
Notes section: 2.2.1.1

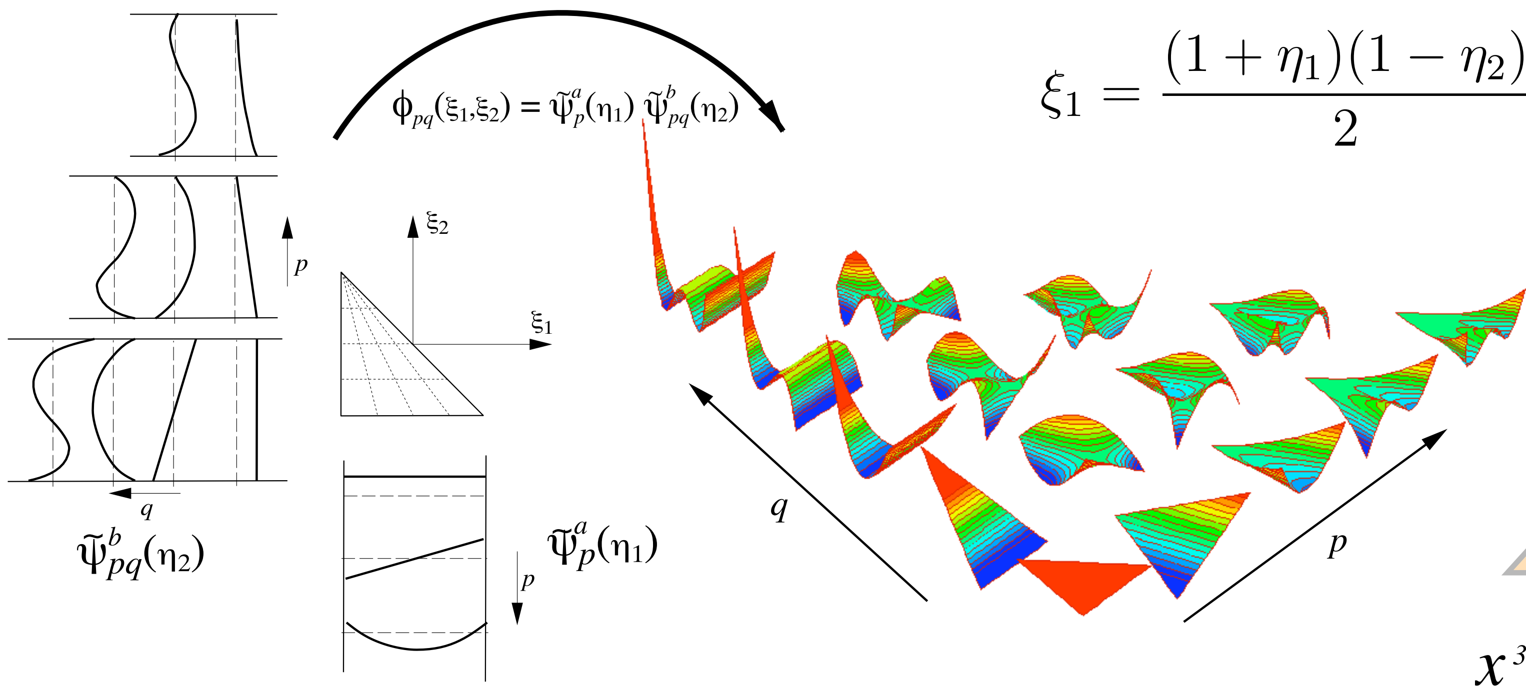
Hybrid element coordinate systems



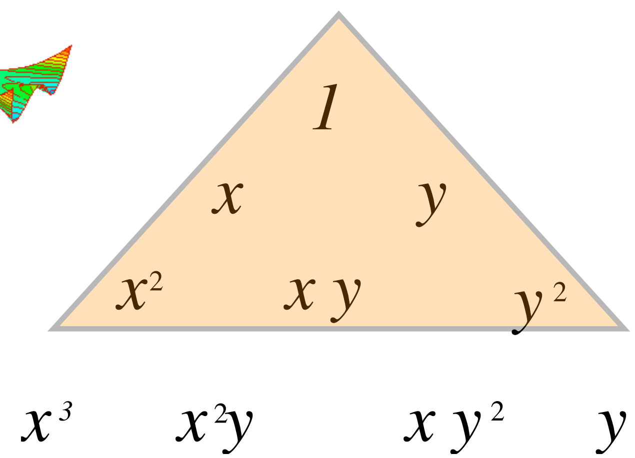


Notes section: 2.2.1.2

Orthogonal Expansion



$$\xi_1 = \frac{(1+\eta_1)(1-\eta_2)}{2} - 1\,, \qquad \xi_2 = \eta_2\,.$$



Principal functions:

$$\widetilde{\psi}^a_p(z) = P^{0,0}_p(z), \qquad \widetilde{\psi}^b_{pq}(z) = \left(\frac{1-z}{2}\right)^p P^{2p+1,0}_q(z)$$

Generalised tensor products:

- Quadrilateral expansion: $\phi_{pq}(\xi_1,\xi_2) = \widetilde{\psi}^a_p(\xi_1)\widetilde{\psi}^a_q(\xi_2), \qquad 0 \le p, q, \le P$
- Triangular expansion: $\phi_{pq}(\xi_1,\xi_2) = \widetilde{\psi}^a_p(\eta_1)\widetilde{\psi}^b_{pq}(\eta_2)\,, \qquad 0 \le p, p+q \le P,$

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Notes section: 2.2.2.1

PKD Orthogonal

Principal functions:

$$\widetilde{\psi}^a_p(z) = P^{0,0}_p(z), \qquad \widetilde{\psi}^b_{pq}(z) = \left(\tfrac{1-z}{2}\right)^p P^{2p+1,0}_q(z)$$

- Triangular expansion: $\phi_{pq}(\xi_1,\xi_2) = \widetilde{\psi}^a_p(\eta_1)\widetilde{\psi}^b_{pq}(\eta_2)$

$$\int_{\mathcal{T}_2} \phi_{ij}(\xi_1,\xi_2)\phi_{pq}(\xi_1,\xi_2)d\xi_1 d\xi_2 = \int_{-1}^{1}\int_{-1}^{1} \underline{\tilde{\psi}^a_i(\eta_1)}\,\underline{\tilde{\psi}^b_{ij}(\eta_2)}\,\underline{\tilde{\psi}^a_p(\eta_1)}\,\underline{\tilde{\psi}^b_{pq}(\eta_2)}\frac{1-\eta_2}{2}d\eta_1 d\eta_2$$

$$\int_{-1}^{1} P^{0,0}_i(\eta_1)P^{0,0}_p(\eta_1)d\eta_1 \int_{-1}^{1}\left(\frac{1-\eta_2}{2}\right)^i\left(\frac{1-\eta_2}{2}\right)^p P^{2i+1,0}_j(\eta_2)P^{2p+1,0}_q(\eta_2)\left(\frac{1-\eta_2}{2}\right)d\eta_2$$

$C_{ip}\delta_{ip}$

$C_{jq}\delta_{jq} \quad \text{if} \quad i = p$

$$\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta P^{\alpha,\beta}_n(x)P^{\alpha,\beta}_i(x)\,d\xi = C\delta_{ni}\,,$$

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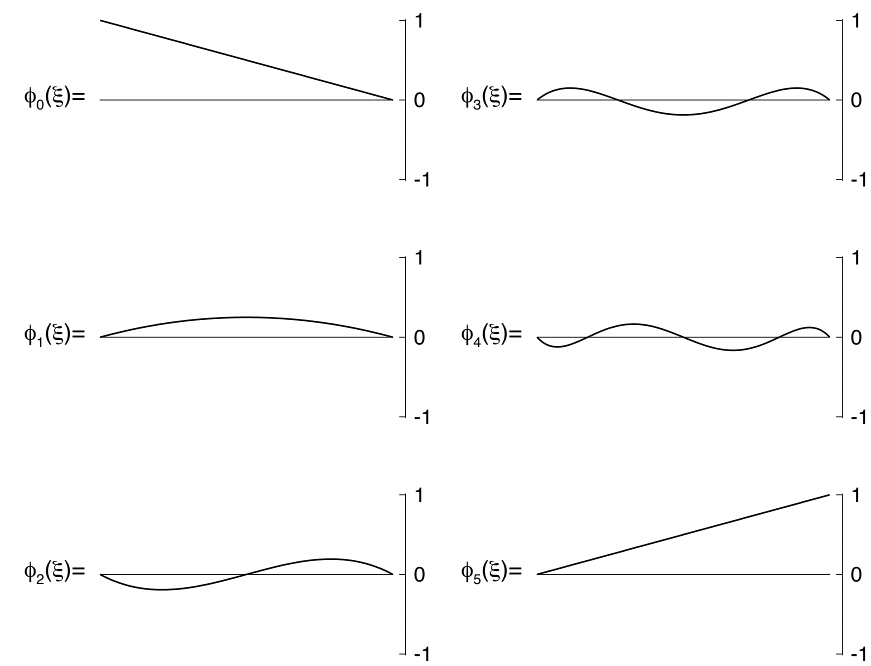
Notes section: 2.2.2.2

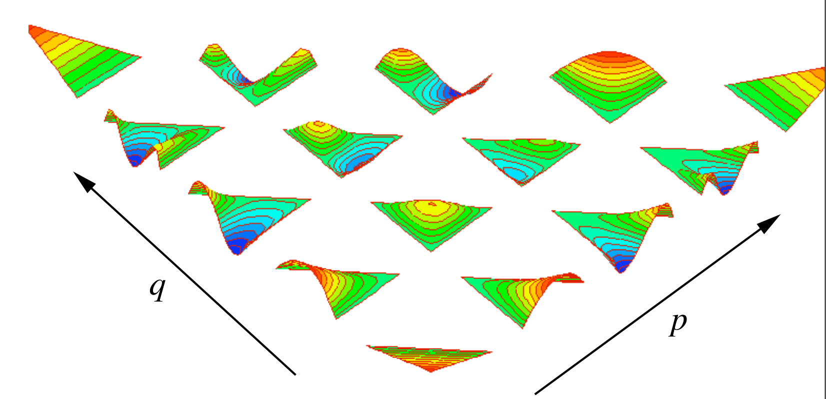
Modified Dubiner Expansion

Modified Principal Functions:

$$\psi_i^a(z) = \begin{cases} \left(\frac{1-z}{2}\right) & i = 0 \\ \left(\frac{1-z}{2}\right)\left(\frac{1+z}{2}\right) P_{i-1}^{1,1}(z) & 1 \le i < I \\ \left(\frac{1+z}{2}\right) & i = I \end{cases} ,$$



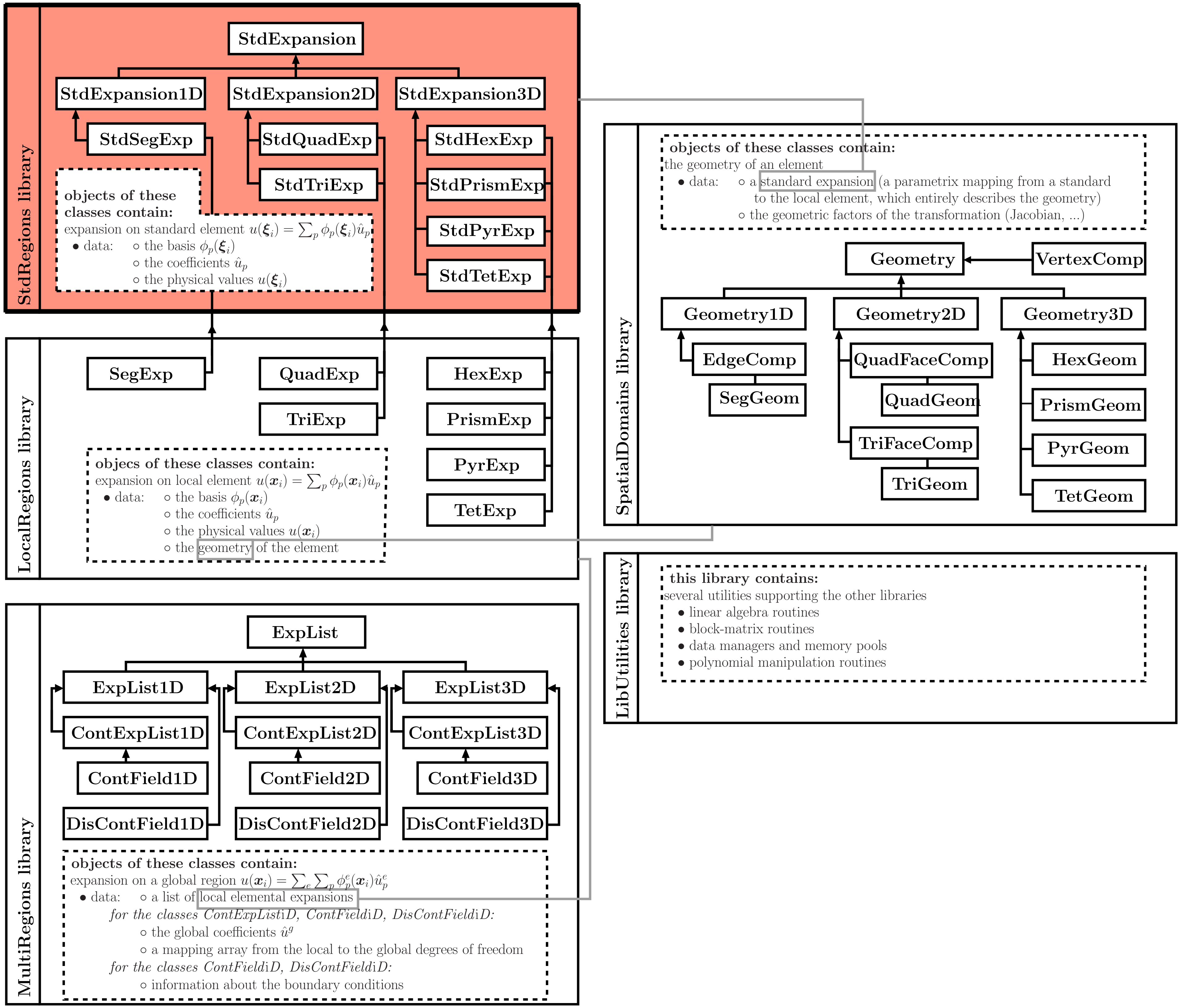
$$\psi_{ij}^b(z) = \begin{cases} \psi_j^a(z) & i = 0, & 0 \le j \le J \\ \left(\frac{1-z}{2}\right)^{i+1} & 1 \le i < I, & j = 0 \\ \left(\frac{1-z}{2}\right)^{i+1}\left(\frac{1+z}{2}\right) P_{j-1}^{2i+1,1}(z) & 1 \le i < I, & 1 \le j < J \\ \psi_j^a(z) & i = I, & 0 \le j \le J \end{cases} ,$$



Generalised tensor products:

- Quadrilateral expansion: $\phi_{pq}(\xi_1, \xi_2) = \psi_p^a(\xi_1)\psi_q^a(\xi_2)\,,$
- Triangular expansion: $\phi_{pq}(\xi_1, \xi_2) = \psi_p^a(\eta_1)\psi_{pq}^b(\eta_2)\,,$

Nektar++ code



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