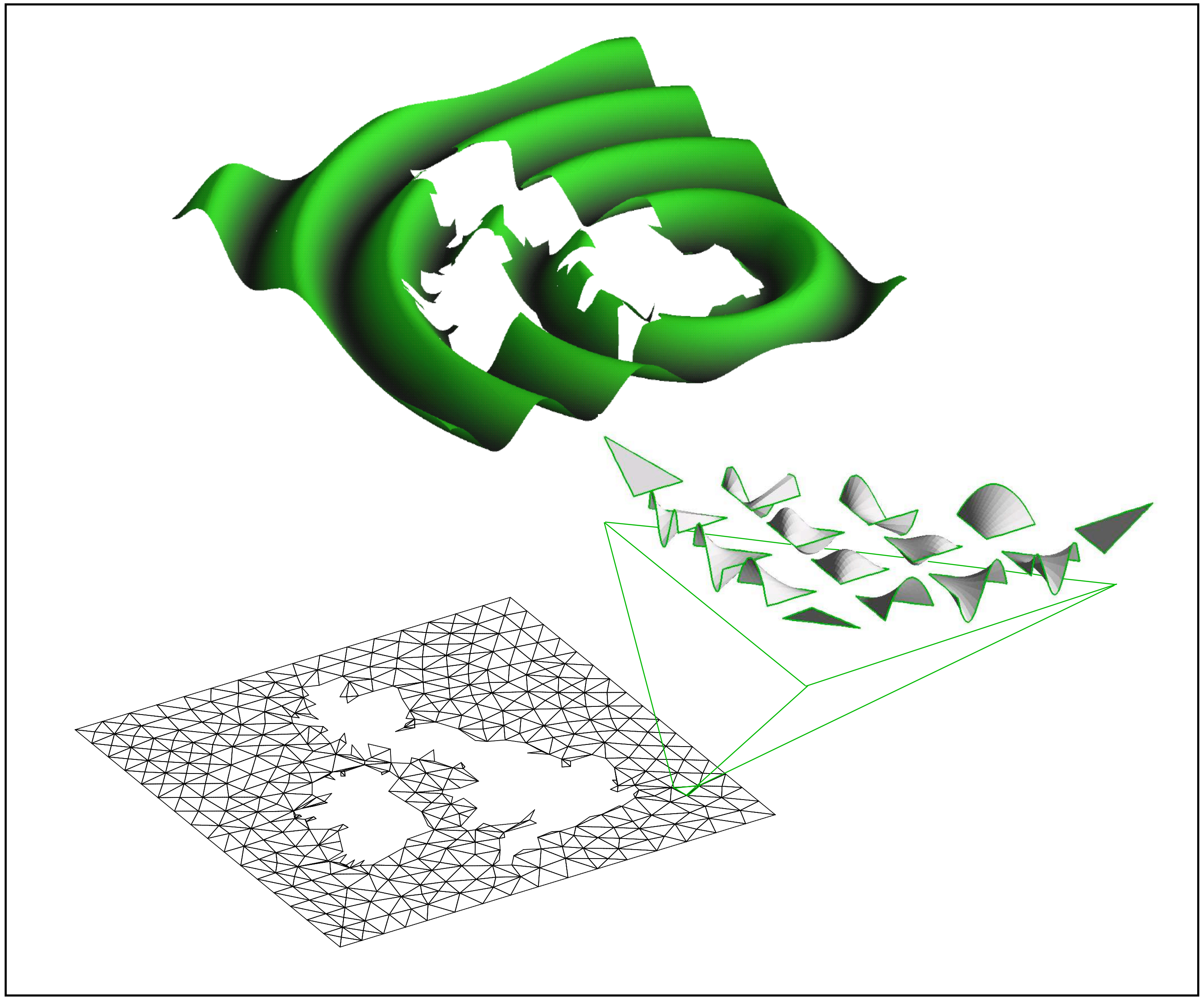
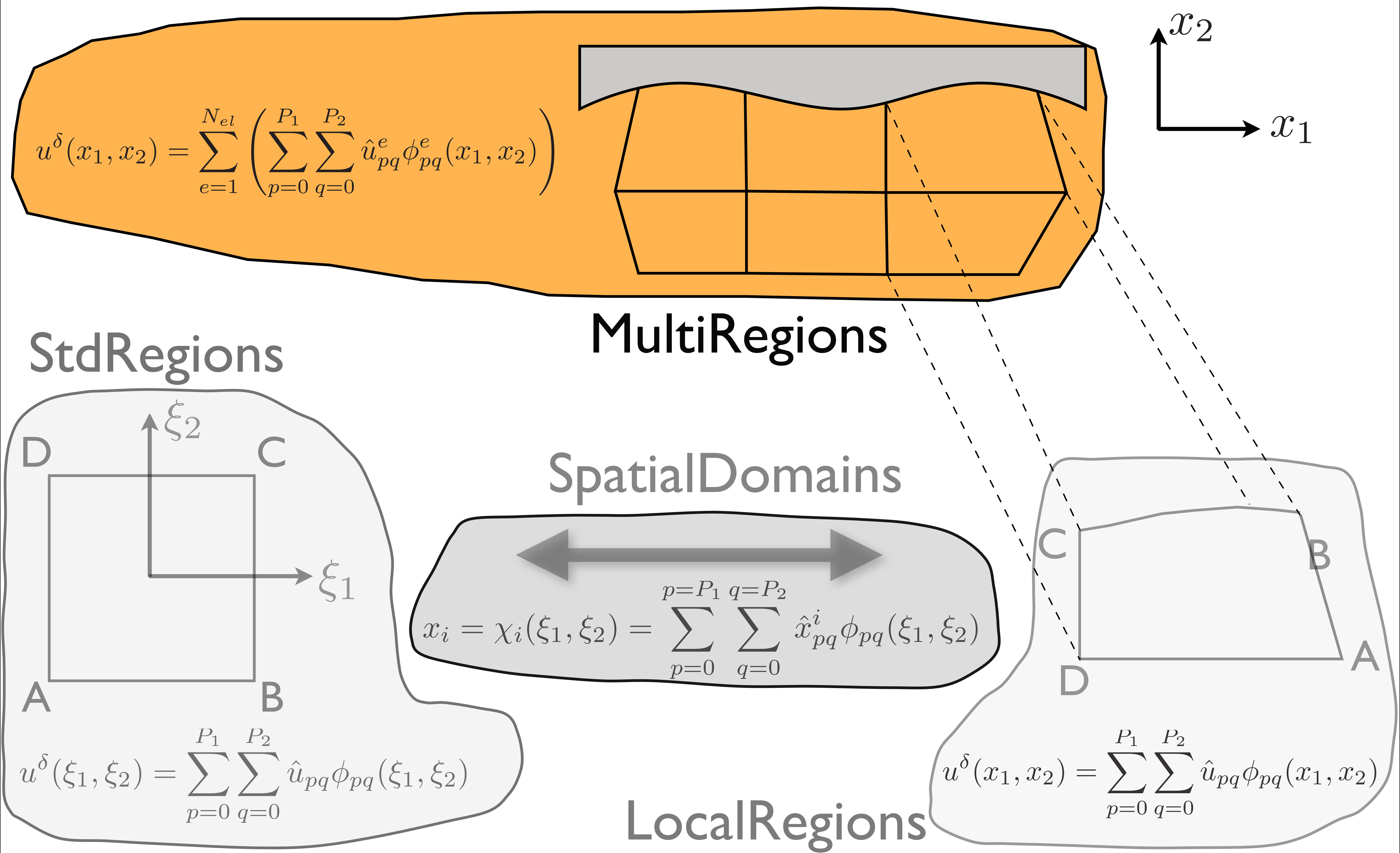
Expansions in Multiple Regions



The big picture



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Outline

- 1D formulation of Helmholtz Problem
  - Elemental formulation
  - Global assembly
- 2D Formulation of Helmholtz problem
  - Global assembly

Helmholtz problem

Poisson Equation: $$\nabla^2 u - \lambda u = f$$

1D: $$\mathbb{L}(u) = \frac{\partial^2 u}{\partial x^2} - \lambda u + f = 0, \qquad u(0) = g_{\mathcal{D}}, \qquad \frac{\partial u}{\partial x}(l) = g_{\mathcal{N}}.$$

Integral formulation (MWR): $$\int_0^l v \frac{\partial^2 u}{\partial x^2} - \int_0^l \lambda v u \; dx + \int_0^l v f \; dx = 0.$$

Integrate by parts: $$\int_0^l v \frac{\partial^2 u}{\partial x^2} dx = \left[v \frac{\partial u}{\partial x} \right]_0^l - \int_0^l \frac{\partial v}{\partial x} \frac{\partial u}{\partial x} dx$$

Weak formulation: $$\int_0^l \frac{\partial v}{\partial x} \frac{\partial u}{\partial x} + \int_0^l \lambda v u \; dx = \int_0^l v f \; dx + \left[v \frac{\partial u}{\partial x} \right]_0^l$$

Enforcing Neumann BC's: $$\Bigg[\underbrace{v(l) \frac{\partial u}{\partial x}\bigg|_l}_{g_{\mathcal{N}}} - \underbrace{v(0) \frac{\partial u}{\partial x}\bigg|_0}_{0} \Bigg]$$

Course notes 1.2

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Helmholtz problem:

Imposing Dirichlet boundary conditions

1D: $\mathbb{L}(u) = \frac{\partial^2 u}{\partial x^2} - \lambda u + f = 0,$ $\boxed{u(0) = g_{\mathcal{D}},}$ $\frac{\partial u}{\partial x}(l) = g_{\mathcal{N}}.$

Weak formulation: $\int_0^l \frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \int_0^l \lambda v u \; dx = \int_0^l v f \; dx + \left[v \frac{\partial u}{\partial x} \right]_0^l$

Lift BC:
(Homogenize problem)

$u^{\delta} = u^{\mathcal{D}} + u^{\mathcal{H}}$ $\boxed{u^{\mathcal{D}}(0) = g_{\mathcal{D}}}$ $u^{\mathcal{H}}(0) = 0$

$$\boxed{\begin{gathered}\int_0^l \frac{\partial v^{\mathcal{H}}}{\partial x}\frac{\partial u^{\mathcal{H}}}{\partial x} dx + \lambda \int_0^l v^{\mathcal{H}} u^{\mathcal{H}} dx = \int v f^* dx \\ f^* = f - v(l) g_{\mathcal{N}} - \frac{\partial v^{\mathcal{D}}}{\partial x}\frac{\partial u^{\mathcal{D}}}{\partial x} + \lambda v^{\mathcal{H}} u^{\mathcal{H}}\end{gathered}}$$

Course notes 1.2

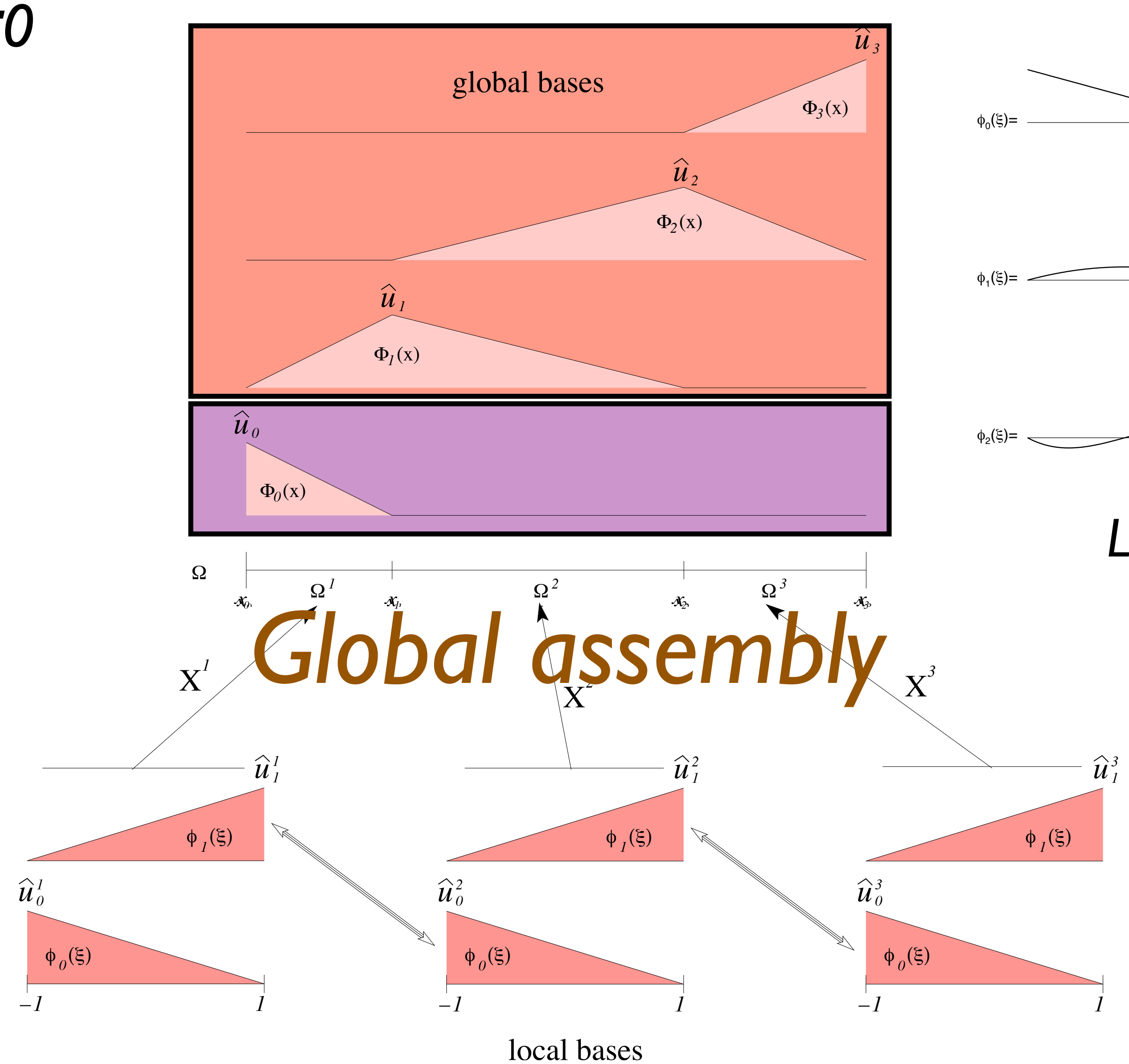
Discrete Approximation

Global approximation - C^0

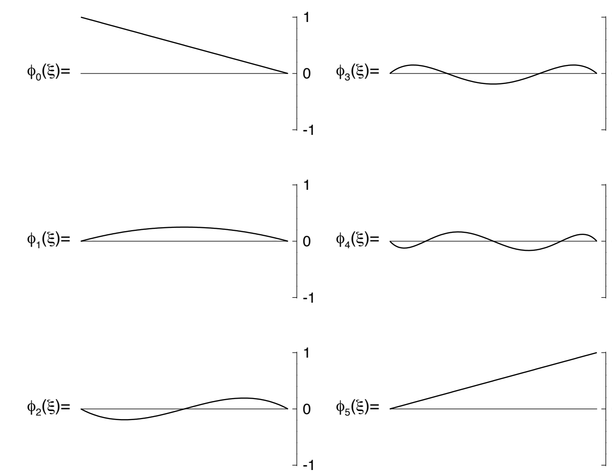
$$u \Rightarrow u^\delta = \sum_i \hat{u}_i \Phi_i(x)$$

$$v \Rightarrow v^\delta = \sum_i \hat{v}_i \Phi_i(x)$$

$$\boxed{u^{\mathcal{D}}(0) = g_{\mathcal{D}}} \qquad \boxed{u^{\mathcal{H}}(0) = 0}$$



Global assembly



Local P-modes

Local approximation

$$u^\delta = \sum_i \hat{u}_i \Phi_i(x) = \sum_e^{nel} \sum_p \hat{u}_p^e \phi_p(x)$$

$$\int_0^l \frac{\partial v^{\mathcal{H}}}{\partial x} \frac{\partial u^{\mathcal{H}}}{\partial x} dx + \lambda \int_0^l v^{\mathcal{H}} u^{\mathcal{H}} dx = \int v f^* dx$$

$$f^* = f - v(l) g_{\mathcal{N}} - \frac{\partial v^{\mathcal{D}}}{\partial x} \frac{\partial u^{\mathcal{D}}}{\partial x} + \lambda v^{\mathcal{H}} u^{\mathcal{H}}$$

Discrete spaces

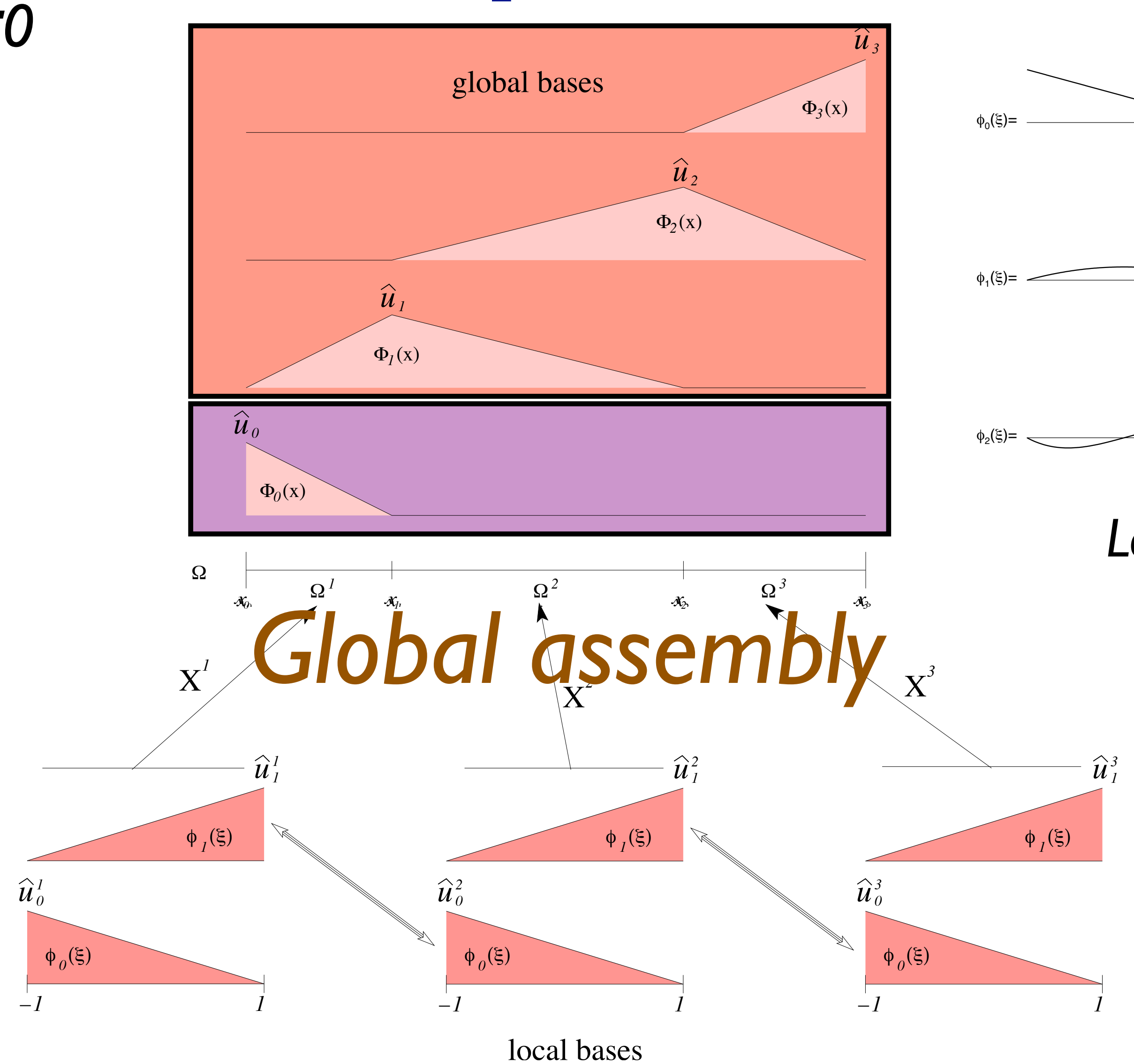
Global approximation - C^0

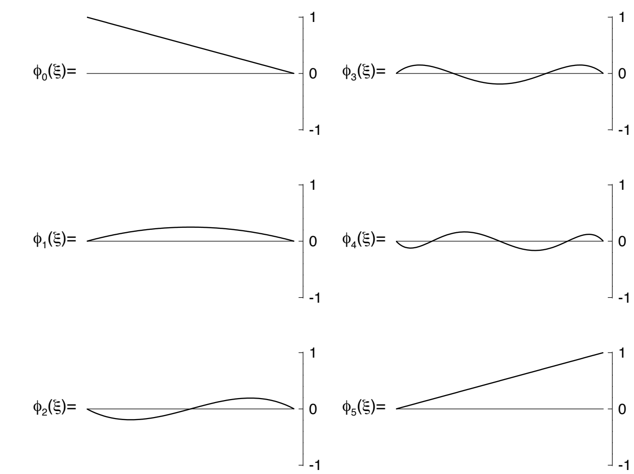
$$u \Rightarrow u^\delta = \sum_i \hat{u}_i \Phi_i(x)$$

$$v \Rightarrow v^\delta = \sum_i \hat{v}_i \Phi_i(x)$$

$$u^{\mathcal{D}}(0) = g_{\mathcal{D}}$$

$$u^{\mathcal{H}}(0) = 0$$





Local P-modes

Global assembly

Local approximation

$$u^\delta = \sum_i \hat{u}_i \Phi_i(x) = \sum_e^{nel} \sum_p \hat{u}_p^e \phi_p(x)$$

$$\sum_i \hat{v}_j \left\{ \sum_j \int_0^l \left[\frac{\partial \Phi_i^{\mathcal{H}}}{\partial x} \frac{\partial \Phi_j^{\mathcal{H}}}{\partial x} + \lambda \Phi_i^{\mathcal{H}} \Phi_j^{\mathcal{H}} \right] \hat{u}_j \, dx = \int \Phi_i^{\mathcal{H}} f^* dx \right\}$$

$$\mathbf{L}[i][j] \qquad \mathbf{M}[i][j] \qquad \mathbf{f}[i]$$

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Outline

- 1D formulation of Helmholtz Problem
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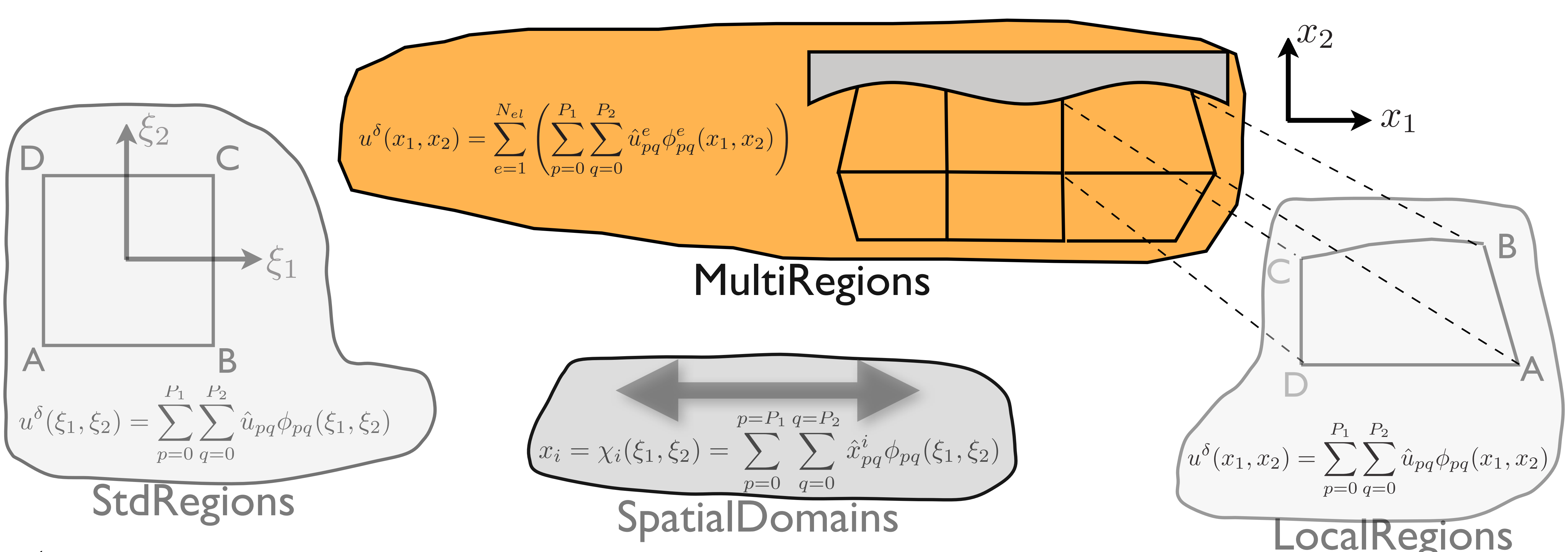
Putting it all together

$$\boxed{\sum_i \hat{v}_j \left\{ \sum_j \int_0^l \left[\frac{\partial \Phi_i^{\mathcal{H}}}{\partial x} \frac{\partial \Phi_j^{\mathcal{H}}}{\partial x} + \lambda \Phi_i^{\mathcal{H}} \Phi_j^{\mathcal{H}} \right] \hat{u}_j \, dx = \int \Phi_i^{\mathcal{H}} f^* dx \right\}}$$

$$\mathbf{f}[i] = \int \Phi_i f^* dx$$



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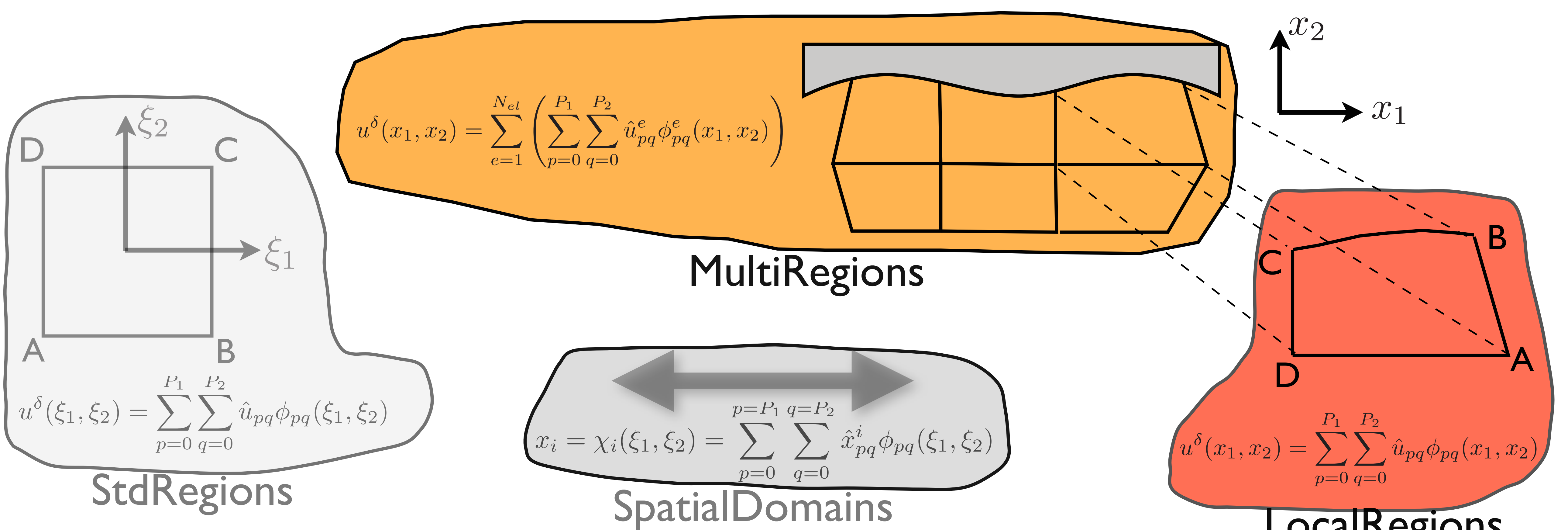
Putting it all together

$$\sum_i \hat{v}_j \left\{ \sum_j \int_0^l \left[\frac{\partial \Phi_i^{\mathcal{H}}}{\partial x} \frac{\partial \Phi_j^{\mathcal{H}}}{\partial x} + \lambda \Phi_i^{\mathcal{H}} \Phi_j^{\mathcal{H}} \right] \hat{u}_j \, dx = \int \Phi_i^{\mathcal{H}} f^* dx \right\}$$

$$\mathbf{f}[i] = \int \Phi_i f^* dx = \sum_e^{nel} \sum_p \int_{\Omega^e} \phi_p(x) f^* dx = \sum_e^{nel} \sum_p \int_{\Omega^e} \phi_p(\chi^e(\xi)) f^* J^e d\xi$$



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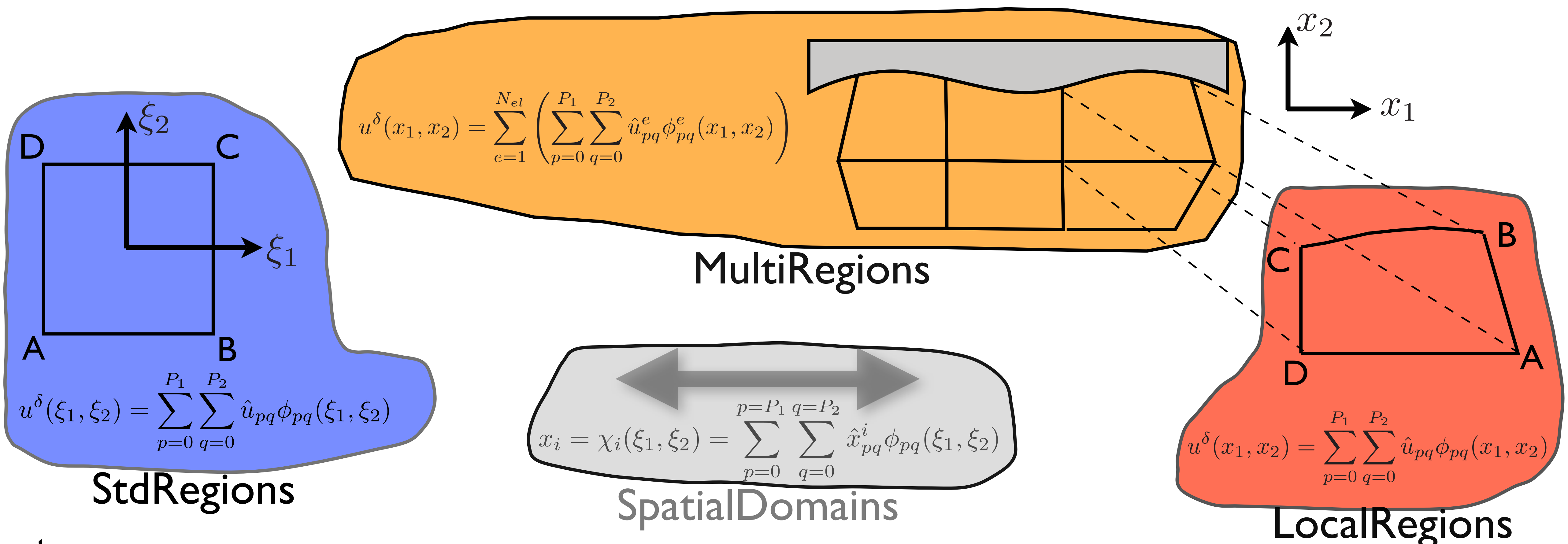
Putting it all together

$$\boxed{\sum_i \hat{v}_j \left\{ \sum_j \int_0^l \left[\frac{\partial \Phi_i^{\mathcal{H}}}{\partial x} \frac{\partial \Phi_j^{\mathcal{H}}}{\partial x} + \lambda \Phi_i^{\mathcal{H}} \Phi_j^{\mathcal{H}} \right] \hat{u}_j \, dx = \int \Phi_i^{\mathcal{H}} f^* dx \right\}}$$

$$\mathbf{f}[i] = \int \Phi_i f^* dx = \sum_e^{nel} \sum_p \int_{\Omega^e} \phi_p(x) f^* dx = \sum_e^{nel} \sum_p \int_{\Omega^e} \phi_p(\chi^e(\xi)) f^* J^e d\xi$$



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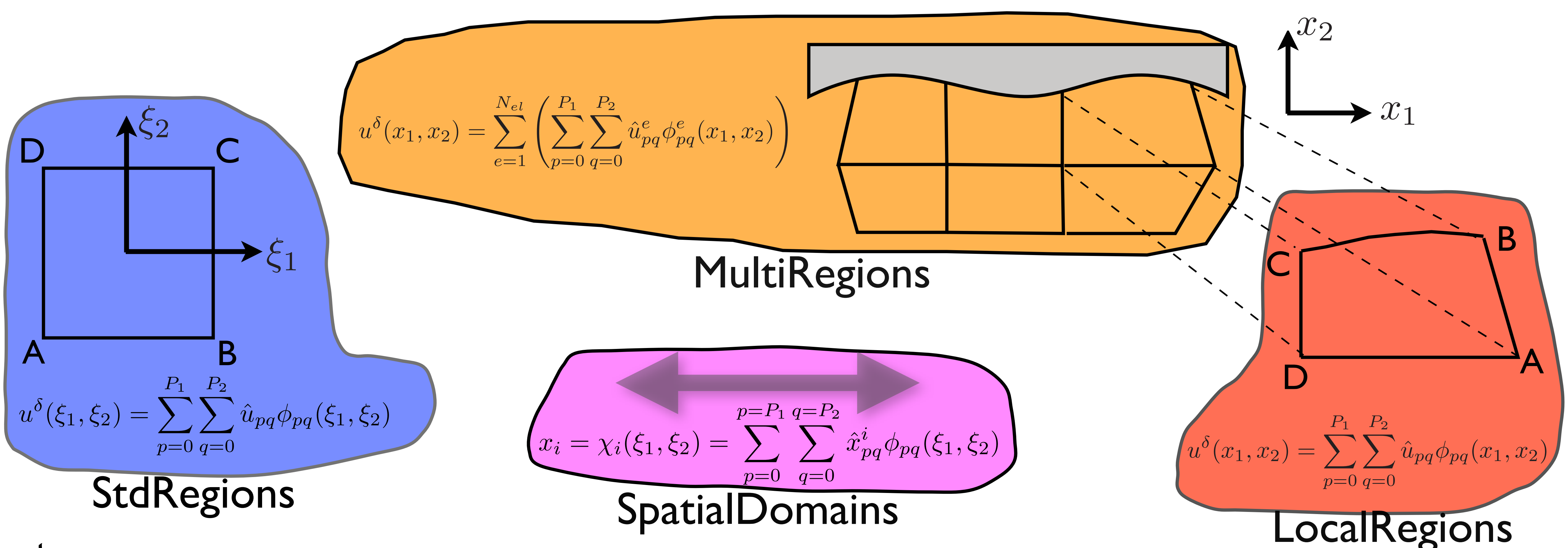
Putting it all together

$$\sum_i \hat{v}_j \left\{ \sum_j \int_0^l \left[\frac{\partial \Phi_i^{\mathcal{H}}}{\partial x}\frac{\partial \Phi_j^{\mathcal{H}}}{\partial x} + \lambda \Phi_i^{\mathcal{H}}\Phi_j^{\mathcal{H}} \right] \hat{u}_j \, dx = \int \Phi_i^{\mathcal{H}} f^* dx \right\}$$

$$\mathbf{f}[i] = \int \Phi_i f^* dx = \sum_e^{nel}\sum_p \int_{\Omega^e} \phi_p(x) f^* dx = \sum_e^{nel}\sum_p \int_{\Omega^e} \phi_p(\chi^e(\xi))\, f^* J^e d\xi$$



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Outline

- 1D formulation of Helmholtz Problem
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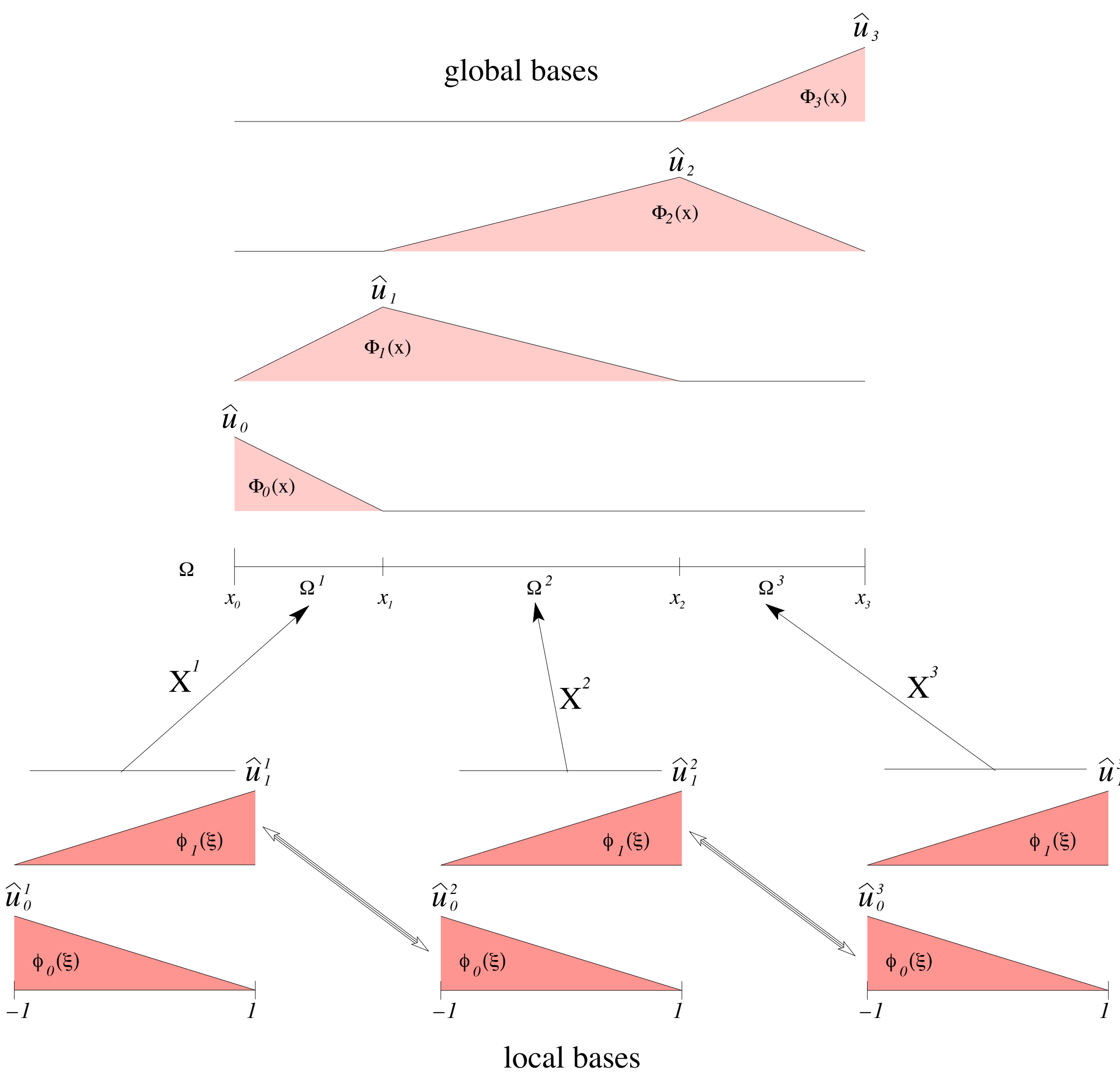
Global Assembly



$$\mathrm{map}[1][i] = \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right\} \qquad \mathrm{map}[2][i] = \left\{ \begin{array}{c} 1 \\ 2 \end{array} \right\} \qquad \mathrm{map}[3][i] = \left\{ \begin{array}{c} 2 \\ 3 \end{array} \right\}$$

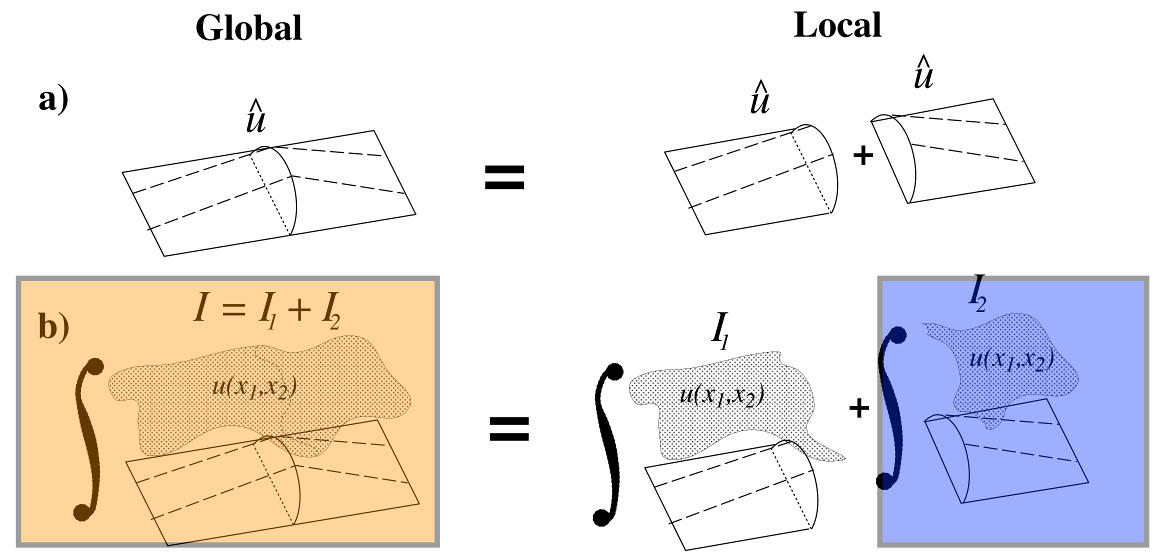


Figure 3.9

$$I^e[i] = \sum_e^{nel} \sum_p \int_{\Omega^e} \phi_p(\chi^e(\xi)) f^* J^e d\xi$$

$$f[i] = \int \Phi_i f^* dx$$

Do $e = 1, N_{el}$
    Do $i = 0, N_m^e - 1$
        $f[\mathrm{map}[e][i]] = f[\mathrm{map}[e][i]] + I^e[i]$
    continue
continue

Course Notes: Sections 1.3.1.4 & 3.2.1

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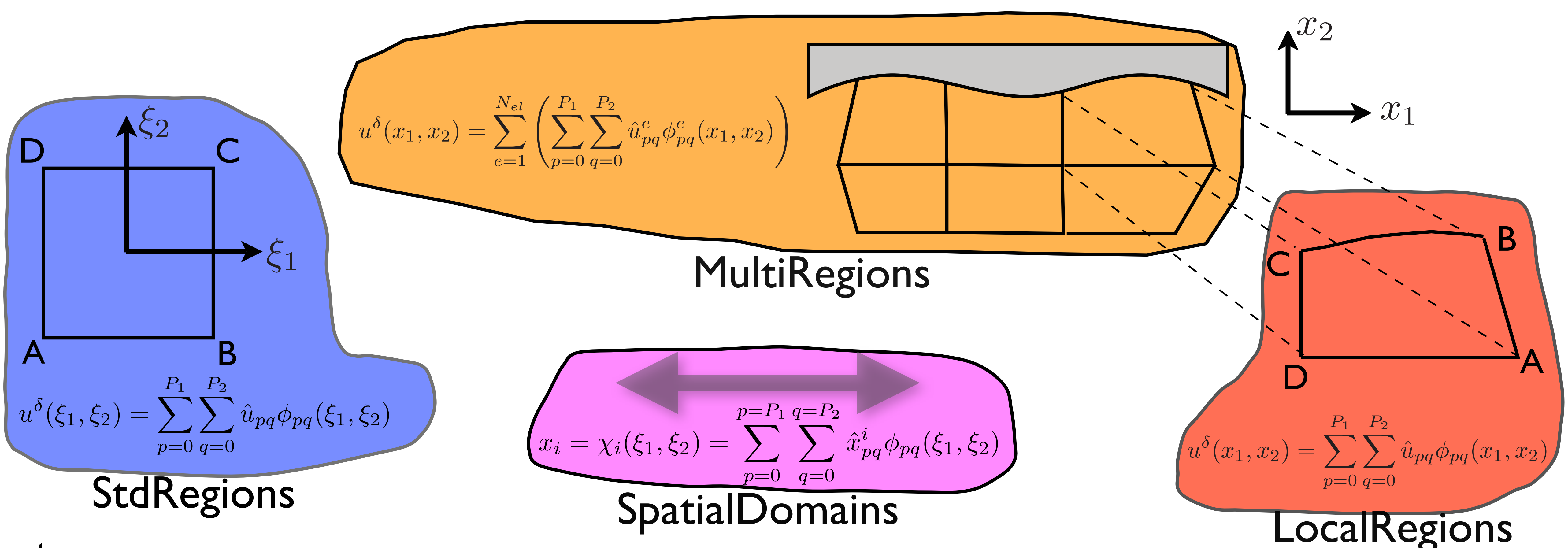
Matrix Construction

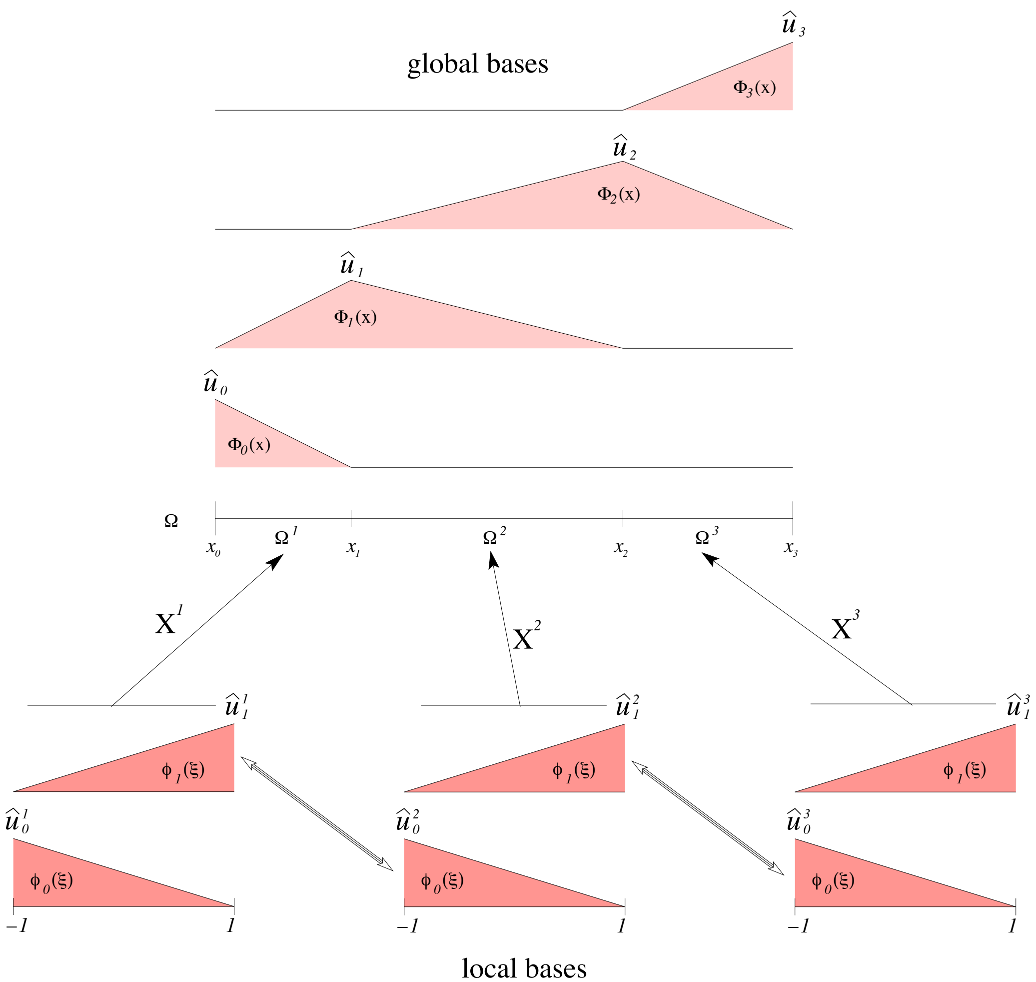
$$\sum_i \hat{v}_j \left\{ \sum_j \int_0^l \left[\frac{\partial \Phi_i^{\mathcal{H}}}{\partial x} \frac{\partial \Phi_j^{\mathcal{H}}}{\partial x} + \lambda \Phi_i^{\mathcal{H}} \Phi_j^{\mathcal{H}} \right] \hat{u}_j \, dx = \int \Phi_i^{\mathcal{H}} f^* dx \right\}$$

$$\mathbf{M}[i][j] = \int_\Omega \Phi_i \Phi_j dx = \sum_e^{nel} \sum_p \sum_q \int_{\Omega^e} \phi_p(x) \phi_q(x) dx$$

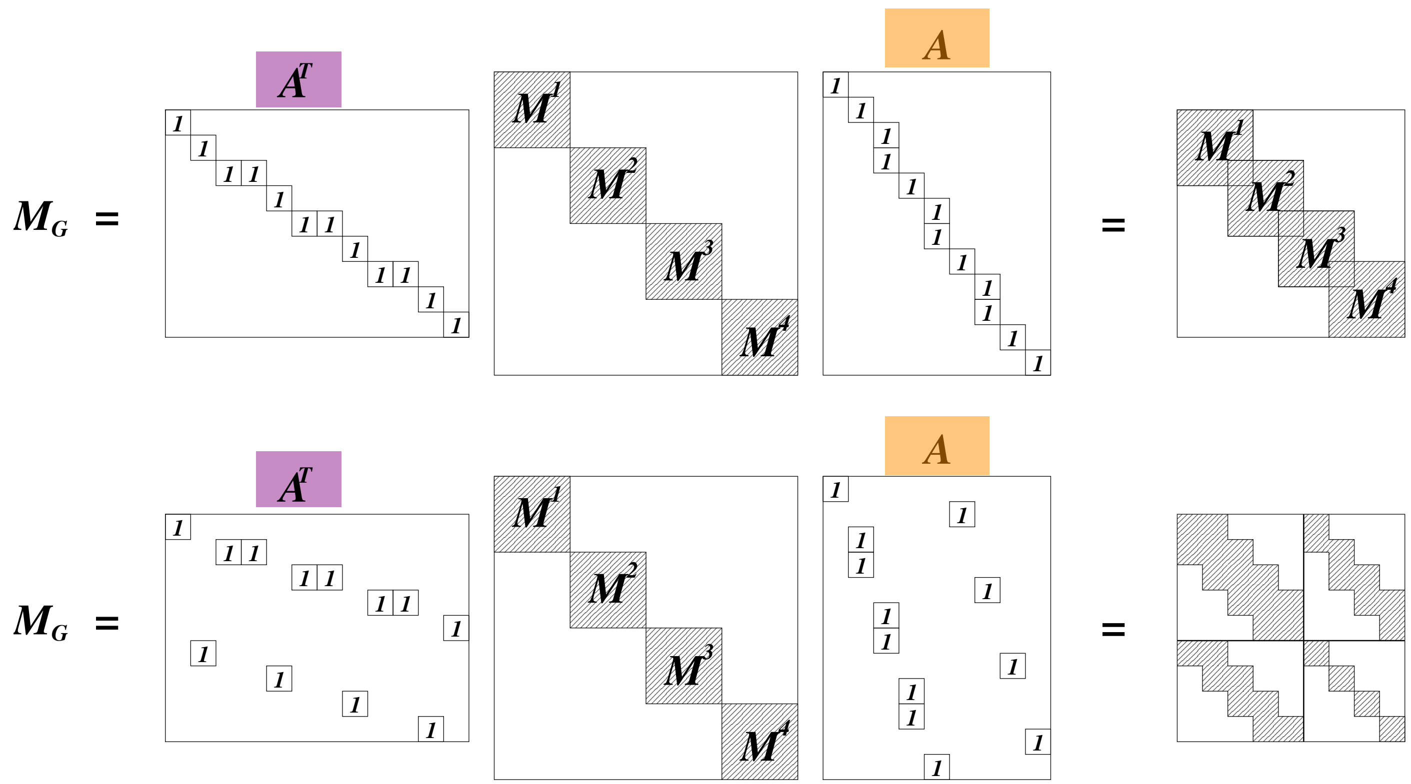
$$= \sum_e^{nel} \sum_p \sum_q \int_{-1}^{1} \phi_p(\chi^e(\xi)) \phi_q(\chi^e(\xi)) j^e d\xi$$



Matrix construction



$$\mathrm{map}[1][i] = \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right\} \qquad \mathrm{map}[2][i] = \left\{ \begin{array}{c} 1 \\ 2 \end{array} \right\} \qquad \mathrm{map}[3][i] = \left\{ \begin{array}{c} 2 \\ 3 \end{array} \right\}$$



$$\left.\begin{array}{l} \text{Do } e = 1, N_{el} \\ \quad \text{Do } i = 0, N_m^e - 1 \\ \qquad \hat{\boldsymbol{u}}^e[i] = \hat{\boldsymbol{u}}_g[\mathrm{map}[e][i]] \\ \quad \text{continue} \\ \text{continue} \end{array}\right\} \Leftrightarrow \hat{\boldsymbol{u}}_l = \boldsymbol{\mathcal{A}}\hat{\boldsymbol{u}}_g,$$

$$\left.\begin{array}{l} \text{Do } e = 1, N_{el} \\ \quad \text{Do } i = 0, N_m^e - 1 \\ \qquad \hat{\boldsymbol{u}}_g[\mathrm{map}[e][i]] = \hat{\boldsymbol{u}}_g[\mathrm{map}[e][i]] + \hat{\boldsymbol{u}}^e[i] \\ \quad \text{continue} \\ \text{continue} \end{array}\right\} \Leftrightarrow \hat{\boldsymbol{u}}_g = \boldsymbol{\mathcal{A}}^T\hat{\boldsymbol{u}}_l$$

Outline

- 1D formulation of Helmholtz Problem
  - Elemental formulation
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- 2D Formulation of Helmholtz problem
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2/3D Helmholtz problem

$$\nabla^2 u - \lambda u = -f$$

Integral formulation:

$$\boxed{\int_\Omega v \nabla^2 u \, d\mathbf{x}} - \int_\Omega \lambda u \, d\mathbf{x} = - \int_\Omega v f \, d\mathbf{x}$$

Divergence theorem:

$$\boxed{\int_\Omega v \nabla^2 u \, d\mathbf{x} = \oint v \frac{\partial u}{\partial n} \, ds - \int_\Omega \nabla v \nabla u \, d\mathbf{x}}$$

Weak form:

$$\int_\Omega \nabla v \nabla u + \lambda u \, d\mathbf{x} = \int_\Omega v f \, d\mathbf{x} + \oint_{\partial\Omega} v \frac{\partial u}{\partial n} \, ds$$

Dirichlet BC: $u = u^{\mathcal{D}} + u^{\mathcal{H}}$

Neumann BC: $\oint_{\partial\Omega} v g_{\mathcal{N}} \, ds$

Tutorial 3, exercise 1

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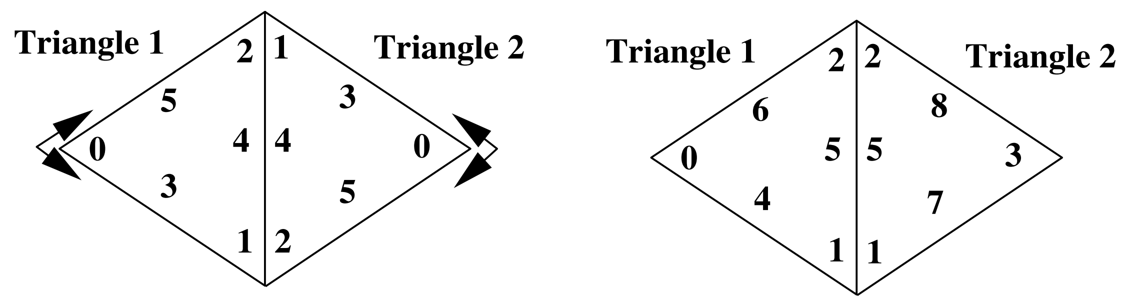
Global assembly



Local Numbering

Global Numbering

$$
\hat{u}_l = \begin{bmatrix} \hat{\boldsymbol{u}}^1[0] \\ \hat{\boldsymbol{u}}^1[1] \\ \hat{\boldsymbol{u}}^1[2] \\ \hat{\boldsymbol{u}}^1[3] \\ \hat{\boldsymbol{u}}^1[4] \\ \hat{\boldsymbol{u}}^1[5] \\ \cdots \\ \hat{\boldsymbol{u}}^2[0] \\ \hat{\boldsymbol{u}}^2[1] \\ \hat{\boldsymbol{u}}^2[2] \\ \hat{\boldsymbol{u}}^2[3] \\ \hat{\boldsymbol{u}}^2[4] \\ \hat{\boldsymbol{u}}^2[5] \end{bmatrix} = \begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & & & & 1 & & & & \\ & & & & & 1 & & & \\ & & & & & & 1 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & 1 & & & & & \\ & & 1 & & & & & & \\ & 1 & & & & & & & \\ & & & & & & & & 1 \\ & & & & 1 & & & & \\ & & & & & & & 1 & \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{u}}_g[0] \\ \hat{\boldsymbol{u}}_g[1] \\ \hat{\boldsymbol{u}}_g[2] \\ \hat{\boldsymbol{u}}_g[3] \\ \hat{\boldsymbol{u}}_g[4] \\ \hat{\boldsymbol{u}}_g[5] \\ \hat{\boldsymbol{u}}_g[6] \\ \hat{\boldsymbol{u}}_g[7] \\ \hat{\boldsymbol{u}}_g[8] \end{bmatrix}.
$$

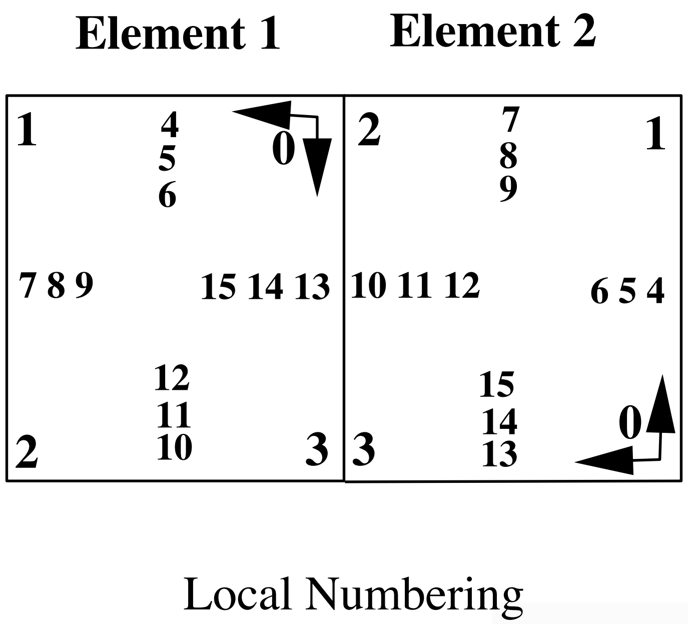
$$
\mathrm{map}[1][i] = \left\{ \begin{array}{c} 0 \\ 1 \\ 2 \\ 4 \\ 5 \\ 6 \end{array} \right. \qquad \mathrm{map}[2][i] = \left\{ \begin{array}{c} 3 \\ 2 \\ 1 \\ 8 \\ 5 \\ 7 \end{array} \right. .
$$

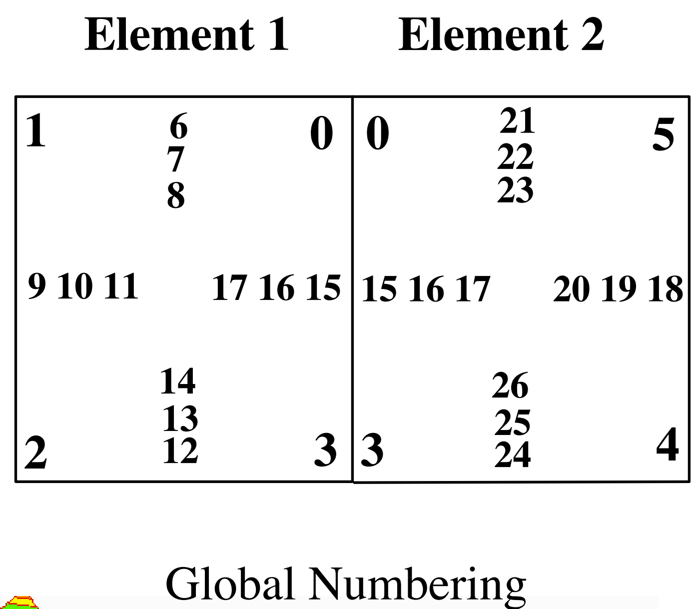
$$
\left.\begin{array}{l}
\text{Do } e = 1, N_{el} \\
\quad \text{Do } i = 0, N_m{}^e - 1 \\
\quad\quad \hat{\boldsymbol{u}}^e[i] = \boxed{\mathrm{sign}[e][i]} \cdot \hat{\boldsymbol{u}}_g[\mathrm{map}[e][i]] \\
\quad \text{continue} \\
\text{continue}
\end{array}\right\} \Leftrightarrow \hat{\boldsymbol{u}}_l = \boldsymbol{\mathcal{A}}\hat{\boldsymbol{u}}
$$

$$
\left.\begin{array}{l}
\text{Do } e = 1, N_{el} \\
\quad \text{Do } i = 0, N_m{}^e - 1 \\
\quad\quad \hat{\boldsymbol{I}}_g[\mathrm{map}[e][i]] = \hat{\boldsymbol{I}}_g[\mathrm{map}[e][i]] \\
\quad\quad\quad\quad + \boxed{\mathrm{sign}[e][i]} \cdot \hat{\boldsymbol{I}}^e[i] \\
\quad \text{continue} \\
\text{continue}
\end{array}\right\} \Leftrightarrow \hat{\boldsymbol{I}}_g = \boldsymbol{\mathcal{A}}^T \hat{\boldsymbol{I}}_l.
$$

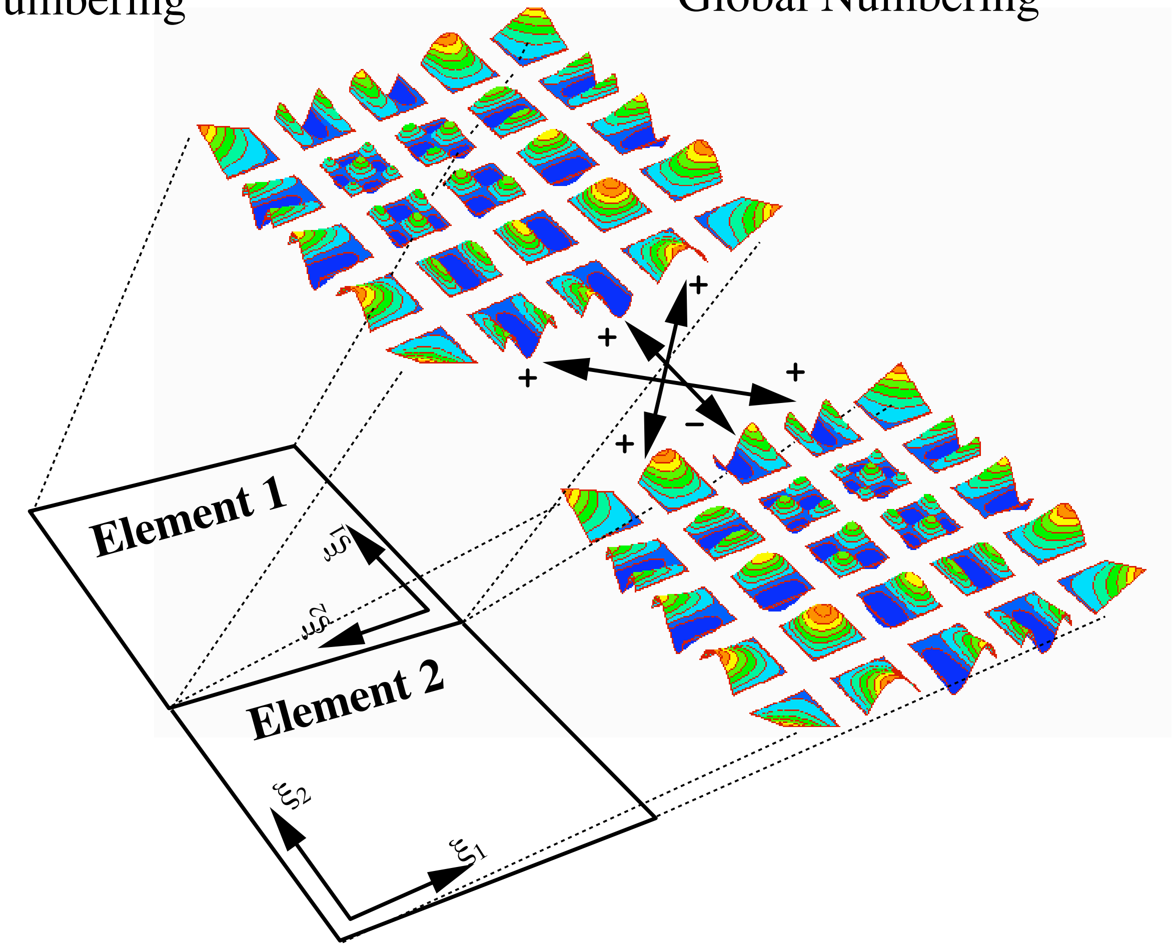
Course Notes: Section 3.2.1

2D Global Assembly: Modal



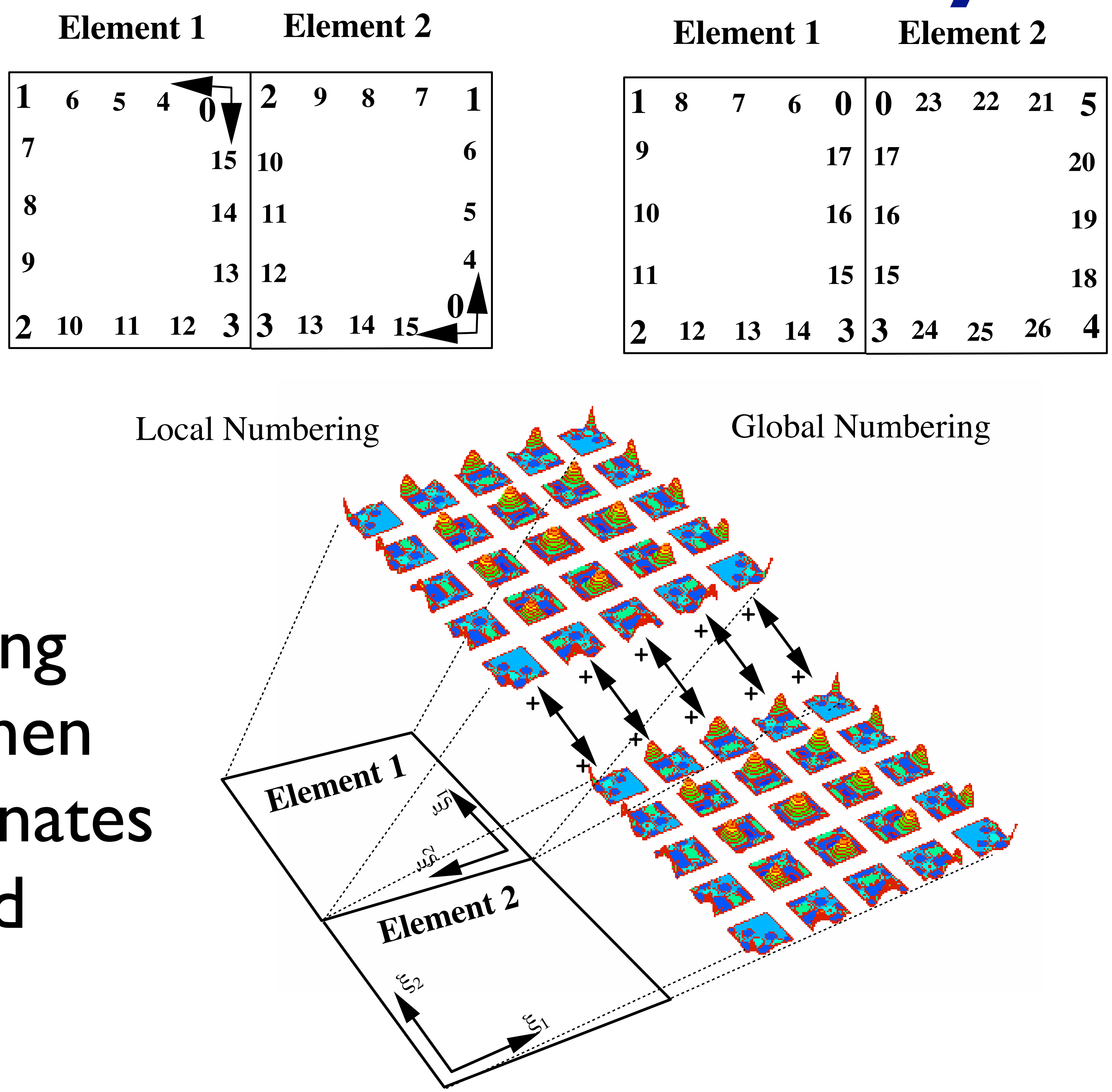


Sign change required when local coordinates reversed



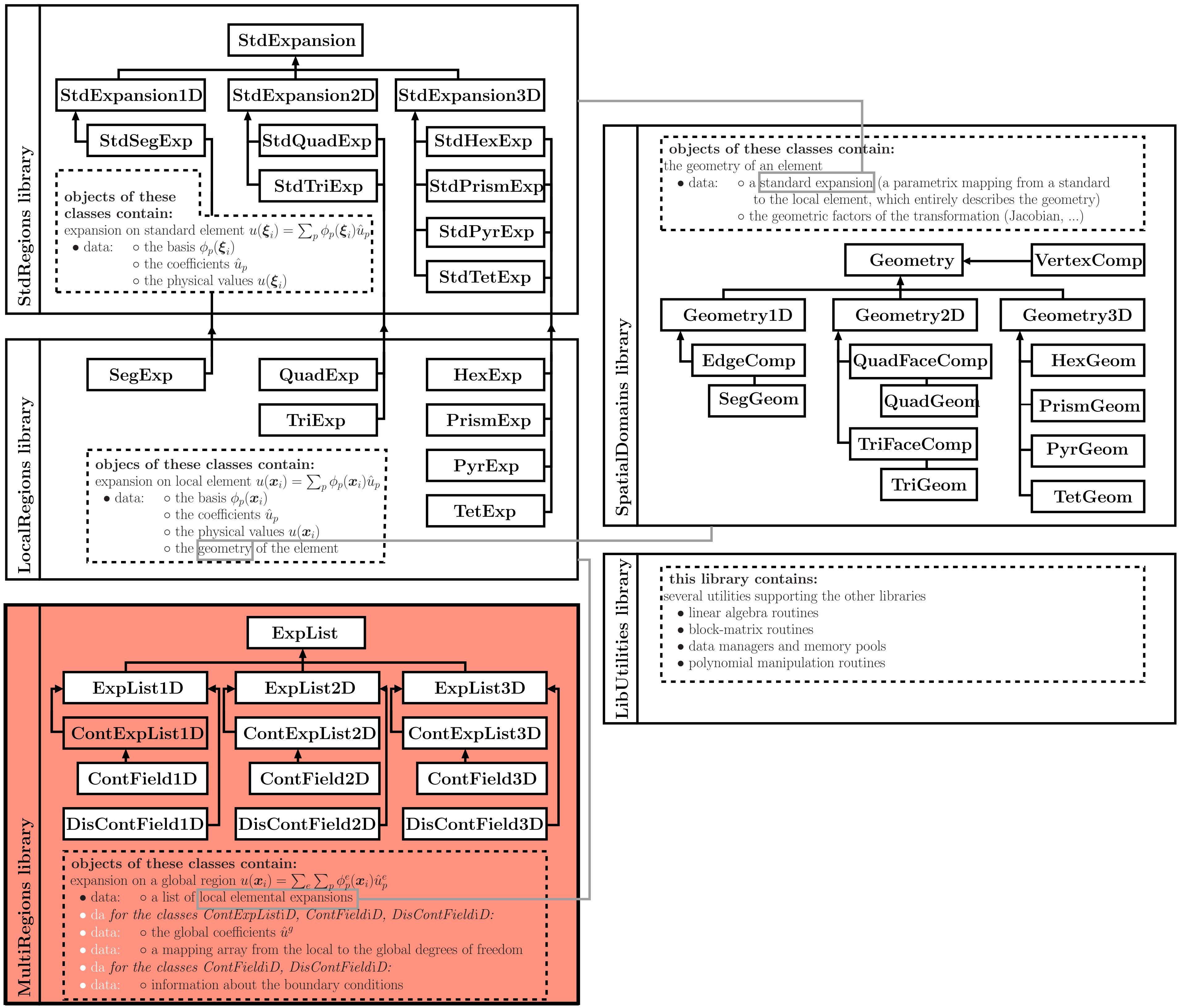
Course Notes: Section 3.2.1.1.

2D Global Assembly: Nodal



Numbering altered when local coordinates reversed

Nektar++ code



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